Sections 5.4: Indefinite Integrals and the Net Change Theorem

In this section we reformulate the idea of antiderivatives to emphasis their relationship with definite integrals.

1. The Indefinite Integral

We start with a definition.

Definition 1.1. We define the definite integral of $f(x)$ and denote it by

$$\int f(x)dx$$

to be the family of all antiderivatives of $f(x)$.

Note that we have not really introduced anything new in this definition - we have simply given a name and added some notation to the family of antiderivatives of a function.

Warning. Do not get the definite integral and the indefinite integral confused - the definite integral is a number and the indefinite integral is a family of functions - they are completely different things!

We look at some examples.

Example 1.2. Evaluate the following indefinite integrals:

(i)

$$\int \left(4t + \frac{1}{t}\right)dt.$$

This is a direct consequence of the power rule for integrals and the derivative of a logarithm. Specifically, we have:

$$\int \left(4t + \frac{1}{t}\right)dt = 2t^2 + \ln\left(|t|\right) + C.$$

(ii)

$$\int \left(\sqrt{(x^3)} - \frac{2}{x}\right)dx.$$

Rewriting, and using the same rules as the previous example, we have

$$\int \left(\sqrt{(x^3)} - \frac{2}{x}\right)dx = \int \left(x^{\frac{3}{2}} - \frac{2}{x}\right)dx = \frac{2}{5}x^{\frac{5}{2}} - 2\ln\left(|x|\right) + C.$$

(iii)

$$\int (1+x)^9 dx.$$

At the moment, we have no rule to find the definite integral for a composition like this. However, observe that when we differentiate this function, the "inside" function differentiates to 1 and thus contributes nothing new when we use the chain rule. This suggests we can just apply the regular power rule to $(1+x)^9$ (just as we would to x^9. Specifically,

$$\int (1+x)^9 dx = \frac{(1+x)^{10}}{10} + C.$$

We can check our answer by differentiating.

(iv)

$$\int (t^2 \ln(t) - t^{\ln(t)-t} + t^e e^t) dx.$$

This looks extremely difficult, but observe that the integral is with respect to x and the variable in the integrand is with respect to t. Thus we have

$$\int (t^2 \ln(t) - t^{\ln(t)-t} + t^e e^t) dx = (t^2 \ln(t) - t^{\ln(t)-t} + t^e e^t) x + C.$$

During the second semester of calculus, you will develop many techniques to determine the indefinite integral of many functions. However, there are a small number of indefinite integrals you should always know (see Table 1 on page 393).

2. Evaluating Definite Integrals

As mentioned previously, one of the reasons for expressing antiderivatives in terms of integrals is to emphasis the relationship between the definite integral and antiderivatives given by the fundamental theorem of calculus. To this end, we usually write

$$\int_a^b f(x)dx = \int f(x)dx \Big]_a^b.$$

We illustrate with a couple of examples.

Example 2.1. Evaluate the following definite integrals.

(i)

$$\int_0^2 \left(\frac{x^3}{3} + 2x\right) dx.$$

We have

$$\int_0^2 \left(\frac{x^3}{3} + 2x\right) dx = \left[\frac{x^3}{9} + x^2\right]_0^2 = \frac{8}{9} + 4.$$

(ii)

$$\int_0^{\frac{\pi}{4}} (\sin(t) + \cos(t)) dt.$$

We have

$$\int_0^{\frac{\pi}{4}} (\sin(t)+\cos(t))dt = [-\cos(t)+\sin(t)]_0^{2} = (-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2})-(-1+0) = 1.$$

(*iii*)

$$\int_0^1 3e^x dx.$$

We have

$$\int_0^1 3e^x dx = [3e^x]_0^1 = 3e - 3.$$

3. The Net Change Theorem

FTC part 2 says that if F is an antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This statement can be reinterpreted in terms of derivatives. Specifically, we have the following.

Result 3.1. If $F'(x)$ is the rate of change of $F(x)$, then the definite integral of $F'(x)$ from a to b is equal to the net change of $F(x)$ over $[a, b]$. More precisely, the integral of a rate of change is the net change i.e.

$$\int_a^b F'(x)dx = F(b) - F(a).$$

This can be interpreted in many different ways depending upon what $F(x)$ is measuring. For example, if $v(t)$ is the velocity of a moving object, then since $s'(t) = v(t)$ where $s(t)$ is the distance traveled by the moving object, the integral

$$\int_a^b v(t)dt = s(b) - s(a)$$

is equal to the net change in distance of the object between $t = a$ and $t = b$. For other interpretations, see the text (pages 394 and 395). We finish with an example of the net change theorem.

Example 3.2. Suppose that $v(t) = 3t - 5$ for $0 \leqslant t \leqslant 3$ is the velocity of a moving particle at time t. Answer, the following.

(*i*) Determine the displacement of the particle over the interval $[0, 3]$.

The displacement of the particle will be the net change of the particle over this time period. Thus

$$\text{Displacement } = \int_0^3 (3t-5)dt = \frac{3t^2}{2} - 5t\Big|_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}.$$

(*ii*) Determine the total distance traveled by the particle over the interval $[0, 3]$.

The total distance traveled by the particle will be equal to absolute value of the distance traveled in one direction plus the absolute value of the distance traveled in the other. Thus we first need to determine if the particle ever turns around on this interval. This can only happen if $v(t) = 0$, and this happens when $t = 5/3$. Since $5/3$ is in the interval $[0, 3]$, the particle may turn around, and thus we need to calculate two integrals to determine the total distance traveled. Specifically, we have

$$\begin{aligned} \text{Total Disance } &= \left| \int_0^{\frac{5}{3}} (3t-5)dt \right| + \left| \int_{\frac{5}{3}}^{3} (3t-5)dt \right| \\ &= \left| \frac{3t^2}{2} - 5t \right]_0^{\frac{5}{3}} \Bigg| + \left| \frac{3t^2}{2} - 5t \right]_{\frac{5}{3}}^{3} \Bigg| \end{aligned}$$

$$= \left| \frac{3 \cdot (\frac{5}{3})^2}{2} - 5 \cdot \frac{5}{3} - (0) \right| + \left| \frac{3 \cdot (3^2)}{2} - 5 \cdot (3) - \left(\frac{3 \cdot (\frac{5}{3})^2}{2} - 5 \cdot \frac{5}{3} \right) \right| = 4.16 + 2.66 = 6.82.$$

Note by the previous example, there is a big difference between total distance traveled and net distance. In general, the net change of a function is the difference between the value of the function at the beginning and at the end.