Calculus Review

"If you don't know this, REVIEW!".

• The following is a general review of Calculus I with an emphasis on some of the topics prerequisite to the topics covered in Calculus 2. If you do not recognize something or you have forgotten some of the techniques, **REVIEW NOW** - do not take the ostrich approach as it will be much harder to play catch up later in the semester. We break the review up into three different pieces: Precalculus, differentiation and integration.

PreCalculus

Of course, all of precalculus is prerequisite to any calculus course. However there are some topics which are especially important which you may have forgotten.

(i) Rational Functions: Combining and Reducing (Important for Integration Techniques).

Example 0.1. Reduce the rational function $\frac{x^3}{(x-1)^3}$.

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3 - 3x^2 + 3x - 1 & \quad x^3 \\ & - x^3 + 3x^2 - 3x + 1 \\ \cline{2-2} & \quad 3x^2 - 3x + 1 \end{array}$$

So

$$\frac{x^3}{(x-1)^3} = 1 + \frac{3x^2 - 3x + 1}{(x-1)^3}.$$

Example 0.2. Write $\frac{x}{(x-1)^2} + \frac{x^2}{x^3-1}$ as a single fraction in lowest terms.

Observe that $(x^3 - 1) = (x - 1)(x^2 + x + 1)$ and $(x - 1)^2 = (x - 1)(x - 1))$, so the common denominator will be: $(x - 1)^2(x^2 + x + 1)$. Then we get:

$$\frac{x}{(x-1)^2} + \frac{x^2}{x^3-1} = \frac{x(x^2+x+1) + x^2(x-1)}{(x-1)^2(x^2+x+1)} = \frac{2x^3 + x}{(x-1)^2(x^2+x+1)}.$$

(ii) Completing the Square for a Quadratic: (Important for Integration Techniques)

Suppose that $ax^2 + bx + c$ is a quadratic function (so $a \neq 0$). To find the zeros or show that there are no zeros, we usually apply the quadratic formula. The quadratic formula was derived algebraically using a method called "completing the square" which is critical for certain integration. It is derived as follows:

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a((x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a})$$
$$= a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c.$$

Example 0.3. Complete the square for the following two quadratics and use it to find the zeros or show there are none:

(a) $x^2 + 2x + 3$.

$$x^2 + 2x + 3 = (x + 1)^2 - 1 + 3 = (x + 1)^2 + 2.$$

$(x + 1)^2 + 2 = 0$ implies $(x + 1)^2 = -2$, so there are no solutions (no zeros!).

(b) $2x^2 - 4x$.

$$2x^2 - 4x = 2(x^2 - 2x) = 2((x - 1)^2 - 1) = 2(x - 1)^2 - 2.$$

$2(x - 1)^2 - 2 = 0$ implies $2(x - 1)^2 = 2$, so $(x - 1)^2 = 1$, and $x - 1 = 1$ or $x - 1 = -1$, giving $x = 2$ or $x = 0$.

Basic Area/Volume Formulas for shapes.

Differentiation

Though most of the material in Calculus II is a study of integration, as we have seen before, integration is closely linked to differentiation, so you need a good knowledge of differentiation techniques.

(*i*) The rules of Differentiation:
- (a) Basic Rules (linearity, constant multiple etc.).
- (b) Basic Functions
- (c) The product rule (products).
- (d) The quotient rule (quotients).
- (e) The chain rule (compositions).

Example 0.4. Differentiate the following functions:

(a) xe^{x^2} (Product Rule and Chain Rule).

(b) $\frac{\sin(x)-1}{x^2}$

(*ii*) Methods of Differentiation
- (a) Implicit Differentiation (Important for polar coordinates and parameterization).
- (b) Logarithmic Differentiation.

Example 0.5. Differentiate the following functions:
(i) $xy = x^2$ (Implicit differentiation).
(ii) $\frac{(x+1)x^x(x-1)}{(x+2)(x-5)}$ (Logarithmic Differentiation).

(*iii*) Applications of Differentiation
(a) Linear approximations (Important for Curve length)
(b) Applications in the Sciences (Important for Applications of Integration).

Integration

Since most of the material in Calculus II is a study of integration, there is a lot in this topic we do not yet know. However, there are certain things you should know.

(*i*) Basic Definitions (what are they, what are the differences between them, how do you find them):
(a) Definite integrals.
(b) Antiderivatives.
(c) Indefinite integrals.

(*ii*) Integration of Basic Functions (page 406).

Example 0.6. Evaluate the following:
(a)

$$\int_1^9 \frac{\sqrt{u} - 2u^2}{u} du$$

(b)

$$\int x + e^x dx$$

(*iii*) Integration by Substitution

Example 0.7. Evaluate the following:
(a)

$$\int_1^{10} \frac{x}{x^2 - 4} dx$$

(b)

$$\int \sin(x) \cos(\cos(x)) dx$$

(*iv*) The Fundamental Theorem of Calculus (Part 2, pg 398).
If F is an antiderivative of f then

$$\int_a^b f(x) dx = F(b) - F(a).$$

(v) Riemann Sums and the Definite Integral.

Recall that the definite integral was constructed using Riemann sums. In this course, especially for the applications, we shall be considering Rieman sums and constructing new functions using Riemann sums. They are very important.

Example 0.8. Evaluate the following:

(a) Find the area bounded between the x-axis and the curve $y = x^2$ between 0 and 3 using three subdivisions of a Riemann sum. Make a sketch of the graph and the Riemann sum and determine if it is an underestimate or an overestimate. How many subdivisions need to be used to get within 0.1 of the actual answer?

(b) An oil slick is in the shape of a circle. Its depth a distance m meters from the center is given by the formula $R(m)$ where $0 < m < 25$ Construct a Riemann sum to estimate the volume of the oil in the slick.
Estimate surface area of a ring by the circumference of the ring multiplied by Δm. Then the volume of any ring will be $2\pi M_1 R(M_1)\Delta M_1$. Summing over all rings, we get

$$\sum_{i=1}^{n} 2\pi M_1 R(M_1)\Delta M_1.$$

How can this estimate be made more exact? What tools of calculus could you use to get an exact answer?