

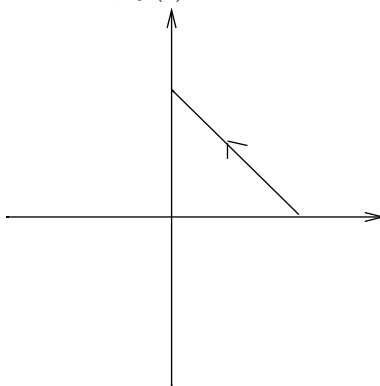
Section 10.1: Curves Defined by Parametric Equations

In the next four sections, we consider new ways to describe curves. We have seen that integration and differentiation can become very complicated when considering x as a function of y or y as a function of x . Also, we have seen that many curves are not defined explicitly as function (like the circle), and require different techniques in order to use calculus (implicit differentiation etc.). By realizing curves in a different way, we can avoid some of these complications. The first new way we consider is realizing a curve using parametric equations.

1. THE BASIC DEFINITION OF PARAMETRIC EQUATIONS

Suppose that a particle P is moving through the plane and at any given time t , the x coordinate is given by the function $x(t)$ and the y coordinate is given by $y(t)$ (so both are functions of time t). Then we call these equations "parametric equations of motion" or just "parametric equations" for the particle. The variable t is called the parameter for the equations. We consider a couple of examples:

Example 1.1. Sketch the curve C traveled by the particle with parametric equations $x(t) = 1 - t$, $y(t) = t$ for $0 \leq t \leq 1$.

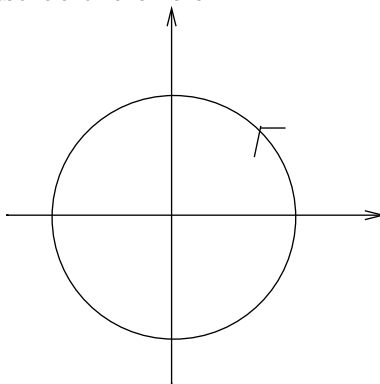


Example 1.2. Sketch the curve C with parametric equations $x(t) = \cos(t)$, $y(t) = \sin(t)$.

In order to sketch this graph, we shall sketch a few points to try to determine a general shape. Observe that when we plug in the values $t = \pi/4, \pi/2, 3\pi/4, \pi$ and other such multiples of π , we get points around a circle. We would like to conclude that in fact the resulting parametric curve is a circle, but just because the points we have looked at lie on the circle doesn't mean this has to be true in general. Therefore, we need to argue why this is the case without reference to specific points. To do this, we make the following observations:

- (i) $\cos(t)$ and $\sin(t)$ are both periodic functions with the same period, so we only need consider $0 \leq t \leq 2\pi$ to determine a complete graph (because it will start repeating after that).
- (ii) Both parametric equations are continuous - this means there can be no jumps on C .
- (iii) Every point lies on the unit circle: $x(t)^2 + y(t)^2 = \cos^2(t) + \sin^2(t) = 1$.
- (iv) $x(t)$ goes from -1 to 1 and $y(t)$ goes from -1 to 1 , so every point must be hit on the circle.

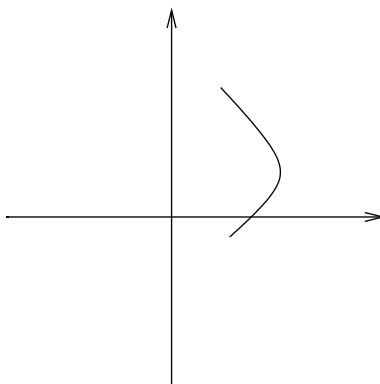
This the curve C must be the circle.



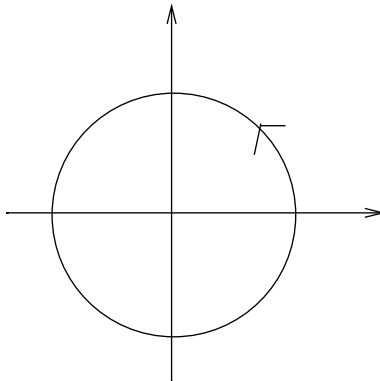
Example 1.3. Sketch the curve C traveled by the particle with parametric equations $x(t) = 1 + \sin(t)$, $y(t) = t - \cos(t)$ for $0 \leq t \leq 2\pi$. We could try a similar approach with this example, but it will probably not work because $y(t)$ is not periodic. Therefore, we make a table of values to try to get an idea of what it looks like.

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$4\pi/3$
x	1	$3/2$	$1 + \sqrt{2}/2$	$1 + \sqrt{3}/2$	2	$1 + \sqrt{3}/2$
y	-1	$\pi/6 - \sqrt{3}/2$	$\pi/4 - \sqrt{2}/2$	$\pi/3 - 1/2$	$\pi/2$	$2\pi/3 + 1/2$
t		$3\pi/4$	$5\pi/6$	π		
x		$1 + \sqrt{2}/2$	$3/2$	1		
y		$3\pi/4 + \sqrt{2}/2$	$5\pi/6 + \sqrt{3}/2$	$\pi + 1$		

Sketching the points, we get a vague idea of what the curve should look like:



How do we get a more accurate graph? We can use technology! The calculator will do all the tedious calculations and attempt to make as accurate a graph as possible given a certain number of inputs. Parametric equations will work on **all** calculators! Using a calculator, we get the following graph which matches our guess.



Graphing parametric equations and using them in general is an extremely important aspect of the concept of parametric equations. Another important concept is being able to derive parametric equations given information about a particle's path (for example, a verbal description or the graph of the path). To illustrate, for the remainder of this section, we shall look at some very important examples. Before we do this, we need the following useful definition.

Definition 1.4. There are two directions a particle can travel along a path. We call the direction it travels the orientation of the path. If an orientation is already specified, we say the path is oriented.

2. CIRCLES

We can use the fact that we know the parametric equations $x(t) = \cos(t)$ and $y(t) = \sin(t)$ define a circle of radius 1 centered at the origin to determine the general parametric equations of a circle. A general circle will have radius R with center at the point (a, b) and will be oriented in either the clockwise or the anticlockwise direction and can start from any point on the circle.

- (i) First, a circle center $(0, 0)$ and radius 1 oriented counterclockwise has parametric equations $\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$.
- (ii) To change the radius to R , we need to increase the values of x and y by R . Thus we will have $\begin{cases} x(t) = R \cos(t) \\ y(t) = R \sin(t) \end{cases}$.
- (iii) To move the center to (a, b) , we need to change the value which x and y oscillate around: $\begin{cases} x(t) = R \cos(t) + a \\ y(t) = R \sin(t) + b \end{cases}$.
- (iv) To change the orientation, we need to change the direction in which it initially travels. This can be done by changing t to $-t$. Observe however that this does change the starting point. $\begin{cases} x(t) = R \cos(-t) + a \\ y(t) = R \sin(-t) + b \end{cases}$.
- (v) To determine the starting point, we just need to modify the value of t via some horizontal shift to make sure it is at the correct place when $t = 0$.

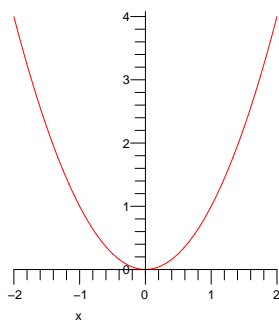
Example 2.1. Find the equation of a circle of radius 2, centered at $(0, 1)$ oriented clockwise and starting at the point $(\sqrt{2}/2, \sqrt{2}/2)$.

We start with the equations $\begin{cases} x(t) = 2 \cos(t) \\ y(t) = 2 \sin(t) + 1 \end{cases}$. In order to make sure it starts off at $(\sqrt{2}/2, \sqrt{2}/2)$, we need to guarantee that when $t = 0$, the value in \cos and \sin is $\pi/4$. Thus, we take $\begin{cases} x(t) = 2 \cos(t + \pi/4) \\ y(t) = 2 \sin(t + \pi/4) + 1 \end{cases}$.

3. FUNCTIONS

Suppose $y = f(x)$ is a function. Finding parametric equations for the graph of $y = f(x)$ is fairly easy, we just use $\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases}$. Alternatively, we could be given the parametric equations for the graph of some function $y = f(x)$ and we may want to write out the equation in cartesian notation (in terms of x and y). We look at a couple of examples to illustrate.

Example 3.1. Write down parametric equations for the curve illustrated below where the orientation is from right to left.



Observe that this is the graph of the function $y = x^2$, so we could try the parametric equations $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$. In this case however, the orientation will not be correct. To change the orientation, we use $x(t) = -t$ (which guarantees that the movement will be from right to left) and substitute $-t$ into y giving $\begin{cases} x(t) = -t \\ y(t) = (-t)^2 = t^2 \end{cases}$.

Example 3.2. Find the Cartesian equation of the curve described by

the parametric equations $\begin{cases} x(t) = \sqrt{t} \\ y(t) = 1 - t \end{cases}$.

For this, we observe that $t = x^2$, so $y = 1 - t = 1 - x^2$. Thus the Cartesian equation will be $y = 1 - x^2$. This can of course be tested on the calculator.

4. ELLIPSES

Observe that the parametric equations $x(t) = A \cos(t)$ and $y(t) = B \sin(t)$ define an ellipse with horizontal radius A and vertical radius B centered at the origin and oriented clockwise. As with circles, we can use this equation to determine a general formula for an ellipse with center at the point (a, b) and oriented in either the clockwise or the anticlockwise direction.

- (i) First, an ellipse center $(0, 0)$ and radii A and B oriented counterclockwise has parametric equations $\begin{cases} x(t) = A \cos(t) \\ y(t) = B \sin(t) \end{cases}$.
- (ii) To move the center to (a, b) , we need to change the values around which x and y oscillate: $\begin{cases} x(t) = A \cos(t) + a \\ y(t) = B \sin(t) + b \end{cases}$.
- (iii) To change the orientation, we need to change the direction in which it initially travels. This can be done by changing t to

$-t$. Observe however that this does change the starting point.

$$\begin{cases} x(t) = A \cos(-t) + a \\ y(t) = B \sin(-t) + b \end{cases} .$$

5. SPEED

Another way we could modify a set of parametric equations is to change the speed at which the particle is moving. The way to do this is through the change $t \rightarrow at$ for some constant a . Performing such an operation is called a **reparametrization**. We make the following observations about reparametrizations:

- (i) If $|a| > 1$, then the particle speeds up.
- (ii) If $|a| < 1$, then the particle slows down.
- (iii) If $a < 0$, the particle changes orientation.
- (iv) If $a > 0$, the orientation remains unchanged.
- (v) If it takes a particle one unit of time to travel its path, we say the particle has unit length.

Example 5.1. Determine the formula for a particle which travels clockwise around a circle of radius 3 centered at $(3, 2)$ starting at $(0, 2)$ traveling at unit speed.

$$\begin{cases} x(t) = 3 \cos(-2\pi t) + 3 \\ y(t) = 3 \sin(-2\pi t) + 2 \end{cases} .$$