Section 10.2: Calculus with Parametric Equations

Just as with standard Cartesian coordinates, we can develop Calculus for curves defined using parametric equations. We shall apply the methods for Cartesian coordinates to find their generalized statements when using parametric equations instead.

1. Tangents

The first thing we studied in Cartesian coordinates was how to find the slope of a tangent line. We can also do this with parametric equations. Suppose that $x = x(t)$ and $y = y(t)$ are parametric equations for x and y and suppose that y can be expressed as a function of x (so $y = F(x)$ for some function F). Then making the substitution $x = x(t)$, we get $y(t) = F(x(t))$. Differentiating, we need to use the chain rule, and we get $y'(t) = F'(x(t))x'(t)$ or $F'(x) = y'(t)/x'(t)$, provided $x'(t) \neq 0$. Thus we get the equation of the tangent to the curve traced by the parametric equations $x(t)$ and $y(t)$ without having to explicitly solve the equations to find a formula relating x and y. Summarizing, we get:

Result 1.1. If $x(t)$ and $y(t)$ are parametric equations, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

provided

$$\frac{dx}{dt} \neq 0.$$

We illustrate with a couple of examples:

Example 1.2. Derive a formula for the slope of an ellipse with parametric equations $x(t) = A\cos(t)$ and $y(t) = B\sin(t)$ at a point (x, y) (your answer should be in terms of x and y).
To find the slope, we apply the formula:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Observe that $x'(t) = -A\sin(t) = -Ay/B$ and $y'(t) = B\cos(t) = Bx/A$. Thus we get

$$\frac{dy}{dx} = -\frac{Bx/A}{Ay/B} = -\frac{B^2x}{A^2y}.$$

Example 1.3. At time t, the position of a particle moving along a curve is given by $x(t) = e^{2t} - e^{-2t}$ and $y(t) = 3e^{2t} + e^{-2t}$.

(i) Find all values of t at which the curve has horizontal or vertical tangent lines.

(ii) Find dy/dx in terms of t.
(iii) Find

$$\lim_{t\to\infty} dy/dx$$

and discuss what this means about the position of the particle.

(i) We observe that $dx/dt = 2e^{2t} + 2e^{-2t}$ and $dy/dt = 6e^{2t} - 2e^{-2t}$.

There is a vertical tangent line when $dx/dt = 0$, so $dx/dt = 2e^{2t} + 2e^{-2t} = 0$ means $2e^{2t} = -2e^{-2t}$, so $e^{2t} = -e^{-2t}$ or $e^{4t} = -1$ which never happens (so there are no vertical tangent lines).

There is a horizontal tangent line when $dy/dt = 0$ and a vertical tangent when $dx/dt = 0$. $dy/dt = 6e^{2t} - 2e^{-2t} = 0$ means $6e^{2t} = 2e^{-2t}$ or $e^{4t} = 1/3$, so $t = \ln(1/3)/4$ is a solution.

(ii) Solving for dy/dx we get $dy/dx = (6e^{2t} - 2e^{-2t})/(2e^{2t} + 2e^{-2t})$.

(iii) Taking the limit $t \to \infty$, we have

$$\begin{aligned}\lim_{t\to\infty} dy/dx &= \lim_{t\to\infty}(6e^{2t} - 2e^{-2t})/(2e^{2t} + 2e^{-2t})\\ &= \lim_{t\to\infty}(6 - 2e^{-4t})/(2 + 2e^{-4t}) = 3.\end{aligned}$$

This means that as $t \to \infty$, the particle will travel approximately linearly along a line with slope 3.

2. Area

When we first considered integration, we used the definite integral to evaluate area bounded by a curve and the x-axis between two points. If $x(t)$ and $y(t)$ are parametric equations, as with Cartesian Calculus, we can use integration to find the area bounded between them (provided one is an explicit function of the other and the curve is only traveled through once!).

Suppose that $x(t)$ and $y(t)$ are parametric equations for a curve C which is also a curve defined explicitly as a function of x (so $y = F(x)$ for some function x). In order to calculate the area under $y = F(x)$ between a and b, we find $\int_a^b F(x)dx$ (provided $F(x) \geqslant 0$). Suppose that $x(\alpha) = a$ and $x(\beta) = b$. Making the substitution $x = x(t)$, we get $dx = x'(t)dt$ so

$$\int_a^b F(x)dx = \int_\alpha^\beta F(x(t))x'(t)dt = \int_\alpha^\beta y(t)x'(t)dt.$$

We illustrate with an example.

Example 2.1. Find the formula for the area of a circle of radius R.

Observe that half a circle will have parametric equations $x(t) = -R\cos(t)$ and $y(t) = R\sin(t)$ where $0 \leqslant t \leqslant \pi$. Since we are restricting to the upper semi-circle, we get y as a function of x, so we can find the area bounded between the x-axis and the curve. We can then multiply this answer by 2 to get the actual area. Applying the formula, we get

$$\int_0^\pi R\sin(t)R\sin(t)dt = R^2\int_0^\pi \sin^2(t)dt = R^2\int_0^\pi \frac{1}{2} - \frac{\cos(2t)}{2}dt$$

$$= R^2\left[\frac{x}{2} + \frac{\sin(2t)}{4}\right]_0^\pi = \frac{\pi R^2}{2}.$$

Thus we get the area of a circle to be πR^2.

3. Arc Length

We recently derived the arc length formula for cartesian functions. Recall that if $y = f(x)$, then the arc length of the graph of $f(x)$ between $x = a$ and $x = b$ is given by the formula

$$\int_a^b \sqrt{(1 + (f'(x))^2)}dx.$$

If $x(t)$ and $y(t)$ are parametric equations for a curve C which is also the graph of a function, we can modify this equation to obtain an equation for arc length in terms of t. Suppose that $x(\alpha) = a$ and $x(\beta) = b$ and that the curve traversed is only traversed once (so the particle does not turn around at any point). Then making the substitution $x = x(t)$ and $dx = x'(t)dt = (dx/dt)dt$, we get

$$\int_a^b \sqrt{(1 + (f'(x))^2)}dx = \int_\alpha^\beta \sqrt{(1 + ((dy/dt)/(dx/dt))^2}\frac{dx}{dt}dt$$

$$= \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt.$$

In fact, even if C is not the graph of a function, this formula is still valid (the only criterion we need is that the curve is traversed just once).
To illustrate how to use this formula, we look at an example.

Example 3.1. Determine the formula for the circumference of a circle using the parametric equation formula.
We observe that the equations are $x(t) = R\cos(t)$ and $y(t) = R\sin(t)$ for $0 \leqslant t \leqslant 2\pi$. Observe that $x'(t) = -R\sin(t)$ and $y'(t) = R\cos(t)$, so

$$\int_0^{2\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \int_0^{2\pi} \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}dt$$

$$= R^2\int_0^{2\pi} dt = 2\pi R^2.$$

as the original formula gives.

4. Surface Area

Recall that if $y = f(x)$ is a function, then we can calculate the surface area of the surface obtained by rotating the graph of $f(x)$ around the x-axis between $x = a$ and $x = b$ using the formula

$$\int_a^b 2\pi f(x)\sqrt{(1 + (f'(x))^2)}dx.$$

As with the other examples we have considered, the general formula to calculate surface area when using parametric equations can be obtained by simple substitution. In general, if C is a curve with parametric equations $x(t)$ and $y(t)$, then the surface area of the volume of revolution for $\alpha \leqslant t \leqslant \beta$ (provided the equations define a function of either x or y) is

$$\int_\alpha^\beta 2\pi y(t)\sqrt{((\frac{dy}{dt})^2 + (\frac{dx}{dt})^2)}dt.$$

We illustrate with an example.

Example 4.1. Find the surface area of the surface obtained by rotating the region bounded by $x = t^3$ and $y = t^2$ for $0 \leqslant t \leqslant 1$ about the x-axis. We just need to apply the formula. Observe that $dx/dt = 3t^2$ and $dy/dt = 2t$, so

$$\text{Surface Area} = \int_0^1 2\pi t^2\sqrt{(4t^2 + 9t^4)}dt = \int_0^1 2\pi t^3\sqrt{((4 + 9t^2))}dt.$$

Setting $u = 9t^2 + 4$, we have $t^2 = (u - 4)/9$ and $du = 18tdt$ or $dt = (t/18)du$, and when $t = 0$, $u = 4$ and when $t = 1$, $u = 13$. Thus

$$\int_0^1 2\pi t^2\sqrt{(t^2(4 + 9t^2))}dt = \int_4^{13} 2\pi\frac{u - 4}{9 * 18}\sqrt{u}du = \frac{2\pi}{162}\int_4^{13} u^{3/2} - 4u^{1/2}du$$

$$= \frac{2\pi}{162}\left[\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right]_4^{13} = \frac{2\pi}{162}\left[\frac{2}{5}(13)^{5/2} - \frac{8}{3}(13)^{3/2} - \frac{2}{5}(4)^{5/2} + \frac{8}{3}(4)^{3/2}\right]$$