

Section 10.3: Polar Coordinates

The polar coordinate system is another way to coordinatize the Cartesian plane. It is particularly useful when examining regions which are circular.

1. CARTESIAN COORDINATES

Before we discuss the polar coordinate system, we recall how the Cartesian coordinate system is set up.

- (i) Fix a point in the plane - call it the origin.
- (ii) Draw two perpendicular lines through the origin - call the one which moves from left to right the x -axis and the one which moved down to up the y -axis.
- (iii) Label the origin $(0, 0)$. Choose a unit length. Label points on the x -axis as follows - start at the origin and for every unit length moving right, we label by the number of units we have moved, for every unit length moving left, we label by the minus the number of units moved. Label points on the y -axis as follows - start at the origin and for every unit length moving up, we label by the number of units we have moved, for every unit length moving down, we label by the minus the number of units moved.
- (iv) Now suppose that P is a point anywhere in the plane. We say P has coordinates (a, b) if it is a units across the x -axis and b units up the y -axis.
- (v) Observe that every point in the plane has Cartesian coordinates and any Cartesian coordinate uniquely determines a point

The Cartesian coordinate system is very useful, and all Calculus has been built up on this. However, we have seen instances where certain problems are extremely difficult in the Cartesian coordinate system (for example, the calculation of the area of a circle is fairly difficult in Cartesian coordinates). We shall now introduce a new coordinate system which can be used instead of the Cartesian system to solve problems in Calculus.

To set up a general coordinate system, we start by fixing a point in the plane which we shall call the pole (usually this is the origin).

2. POLAR COORDINATES

Now we discuss how to set up the polar coordinate system the same way we did for the Cartesian coordinate system.

- (i) Fix a point in the plane - call it the pole.
- (ii) Draw a single half line out from the pole and call it the polar axis (we usually make it coincide with the x -axis).

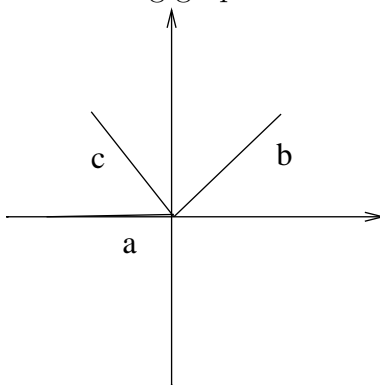
- (iii) Choose a unit length. Label points on the polar axis by starting at the pole and for every unit length moving right, we label by the number of units we have moved
- (iv) Now suppose that P is a point anywhere in the plane which is not the pole. We say P has coordinates (r, ϑ) if it lies a distance r from the origin and makes an angle ϑ with the pole (where angles are measured counter-clockwise).
- (v) We define the pole to be the point with coordinates $(0, \vartheta)$ for any value of ϑ .
- (vi) Observe that any point in the plane has Polar coordinates. However, polar coordinates are not unique because many different coordinates coincide with the same point (which is why we usually restrict to $0 \leq \theta \leq 2\pi$).

We look at a couple of examples.

Example 2.1. Draw the points with polar coordinates:

- (i) $(3, \pi)$
- (ii) $(2, \pi/4)$
- (iii) $(6, 2\pi/4)$

We illustrate all on the following graph:



We make some observations.

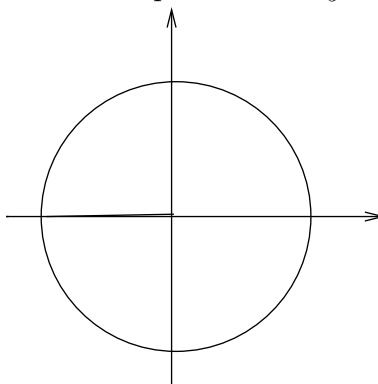
- (i) Observe that if (r, ϑ) are the polar coordinates for a point P , then $(r, \vartheta + 2n\pi + \vartheta)$ and $(-r, (2n + 1)\pi + \vartheta)$ are also coordinates for the point P (WHY?), so every point in the plane has infinitely many different coordinates representing it (which is very different from the Cartesian coordinate system!).
- (ii) We can easily derive the relationship between Cartesian and Polar coordinates. Specifically, if the polar axis coincides with the positive x -axis, then we have $x = r \cos(\vartheta)$ and $y = r \sin(\vartheta)$.
- (iii) With these formulas, we can now deduce formulas relation x and y to r and ϑ . We get $r^2 = x^2 + y^2$ and $\tan(\vartheta) = y/x$.

3. POLAR CURVES

Just as the function $y = f(x)$ defines a graph in the Cartesian plane, we can also consider functions of the radius r as a function of angle ϑ in polar coordinates (so $r = f(\vartheta)$) and their graphs in the Cartesian, or polar plane. More generally, we can consider a polar equation $F(r, \vartheta) = 0$ (as with equations involving x and y) and consider their graphs. To illustrate, we consider some examples:

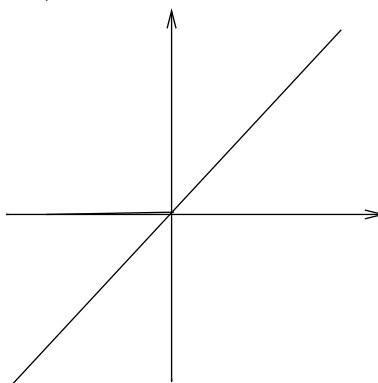
Example 3.1. Sketch the curve $r = 3$ and find a formula for it in terms of Cartesian coordinates.

The curve $r = 3$ will be a circle of radius 3 because there is no restriction on ϑ . This means it will have equation $x^2 + y^2 = 9$.



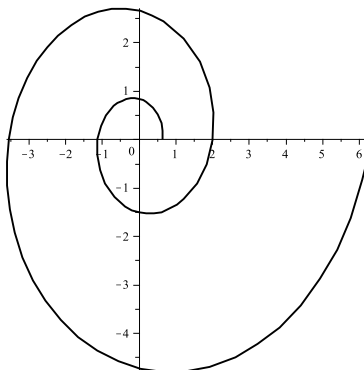
Example 3.2. Sketch the curve $\vartheta = \pi/4$ and find a formula for it in terms of Cartesian coordinates.

Observe that the only restriction is on ϑ , so the graph will be a line which makes an angle $\pi/4$ with the polar axis.



Example 3.3. Sketch the polar curve $r = 2 \cdot (1.2)^\vartheta$.

Observe that this is an exponential function of r in terms of ϑ . Specifically, as ϑ increases, so does r , at a greater and greater rate. In particular, restricting to $-2\pi \leq \vartheta \leq 2\pi$, we have:



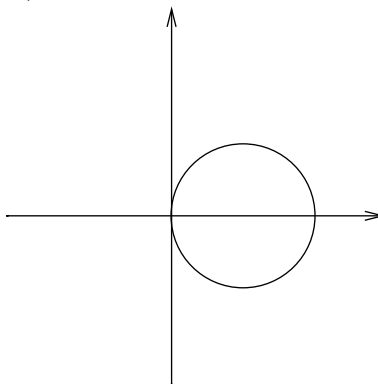
More complicated curves are much harder to sketch.

Example 3.4. Sketch the curve $r = \cos(\vartheta)$ and find an equation for it in terms of Cartesian coordinates.

In order to do this, we could make a table of values and sketch them:

ϑ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1

Observe that when we sketch them, we get points on a circle of radius $1/2$ centered at $(1/2, 0)$.



To prove this for sure, we observe that $r = \cos(\vartheta)$ means that

$$r^2 = r \cos(\vartheta) = x = x^2 + y^2.$$

Thus we have

$$x^2 - x + y^2 = (x - 1/2)^2 - 1/4 + y^2 = 0$$

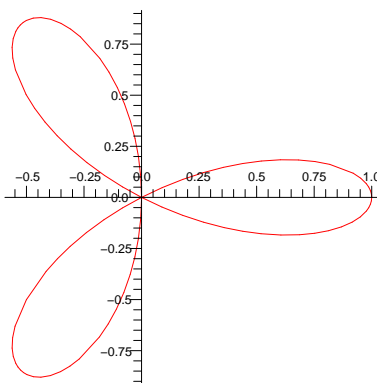
or $(x - 1/2)^2 + y^2 = (1/2)^2$ which is the equation for such a circle.

Example 3.5. Sketch the curve $r = \cos(3\vartheta)$.

This curve is much more difficult because it does not look very easy to convert to Cartesian coordinates. We consider values between 0 and $2\pi/3$ to get a complete graph. We make a table of values.

ϑ	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$	$7\pi/12$	$2\pi/3$
r	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1

Sketching the points, we do not really get a feeling for what this curve should look like. So the question is how do I find a more accurate graph. The answer is to use more points, but of course at some point this is going to get highly frustrating. Thinking back to Cartesian coordinates, how do we usually graph things there? - using Technology. So to find an accurate graph of $r = \cos(3\vartheta)$, we shall use technology. When entered into the calculator, we get the following graph:



Though for more complicated formulas, like the last one, we may have to resort to using a calculator, there are times when we may be able to use symmetry to complete a graph (just like with Cartesian graphs). We make the following observations:

- (i) If we can replace ϑ by $-\vartheta$ without changing the formula, then there is symmetry about the x -axis.
- (ii) If we can replace ϑ by $\pi - \vartheta$ without changing the formula, then there is symmetry about the y -axis.
- (iii) If we can replace ϑ by $\pi + \vartheta$ and r by $-r$ without changing the formula, then there is symmetry about the origin.

Observe that in the last example we considered, the function $r = \cos(3\vartheta)$ had symmetry about the x -axis. If we replace ϑ by $-\vartheta$, then since \cos is an even function, we get $\cos(3\vartheta) = \cos(3(-\vartheta))$ as predicted.

4. TANGENT TO POLAR CURVES

Just as with Cartesian coordinates and parametric equation, we can develop Calculus when using the polar coordinate system. Though we shall wait until the next section to develop integral Calculus, we can consider tangent lines here.

Suppose that $r = f(\vartheta)$ is a polar function. Then using the fact that $x = r \cos(\vartheta)$ and $y = r \sin(\vartheta)$, we get $x = f(\vartheta) \cos(\vartheta)$ and $y = f(\vartheta) \sin(\vartheta)$.

So x and y can be considered as parametric equations in the variable ϑ . By our work on parametric equations, we know how to find the slope of the tangent line when considering parametric equations. Specifically, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\vartheta}}{\frac{dx}{d\vartheta}}.$$

Using the product and chain rule, we have

$$\frac{dx}{d\vartheta} = f'(\vartheta) \cos(\vartheta) - f(\vartheta) \sin(\vartheta) = \frac{dr}{d\vartheta} \cos(\vartheta) - r \sin(\vartheta)$$

and

$$\frac{dy}{d\vartheta} = f'(\vartheta) \sin(\vartheta) + f(\vartheta) \cos(\vartheta) = \frac{dr}{d\vartheta} \sin(\vartheta) + r \cos(\vartheta)$$

so

$$\frac{dy}{dx} = \frac{\frac{dr}{d\vartheta} \sin(\vartheta) + r \cos(\vartheta)}{\frac{dr}{d\vartheta} \cos(\vartheta) - r \sin(\vartheta)}.$$

This is a fairly complicated formula (much more complicated than Cartesian coordinates)!

Example 4.1. Find the points where there is a horizontal tangent lines to $r = \cos(\vartheta)$.

We calculate,

$$\frac{dr}{d\vartheta} = -\sin(\vartheta),$$

so

$$\begin{aligned} \frac{dx}{dy} &= \frac{-\sin(\vartheta) \sin(\vartheta) + \cos(\vartheta) \cos(\vartheta)}{-\sin(\vartheta) \cos(\vartheta) - \cos(\vartheta) \sin(\vartheta)} = \frac{\cos^2(\vartheta) - \sin^2(\vartheta)}{-2 \sin(\vartheta) \cos(\vartheta)} \\ &= \frac{\cos(2\vartheta)}{-\sin(2\vartheta)} = -\cot(2\vartheta). \end{aligned}$$

Horizontal tangent lines occur when $dx/dy = 0$, so when $\tan(2\vartheta)$ is undefined. This means $\vartheta = \pi/4, 3\pi/4$ and any other such value plus or minus a multiple of π .