Section 10.4: Areas and Lengths in Polar Coordinates

In this section, we develop other results from Calculus in the Cartesian plane to Calculus using polar coordinates.

1. Area

Before we develop calculus for polar coordinates, we need to review a couple of formulas for basic trigonometry.

Result 1.1. Suppose S is a sector of a circle of radius R and angle ϑ (measured in radians). Then the area of S is given by the formula $A = r^2\vartheta/2$.

Proof. Observe that the number of radians in a whole circle is 2π and the area of a circle of radius r is πr^2. A sector with angle ϑ represents a fraction of a circle, so the ratio of the area of a segment and the whole circle will be equal to the ratio of the whole angle in a circle and the angle of the segment:

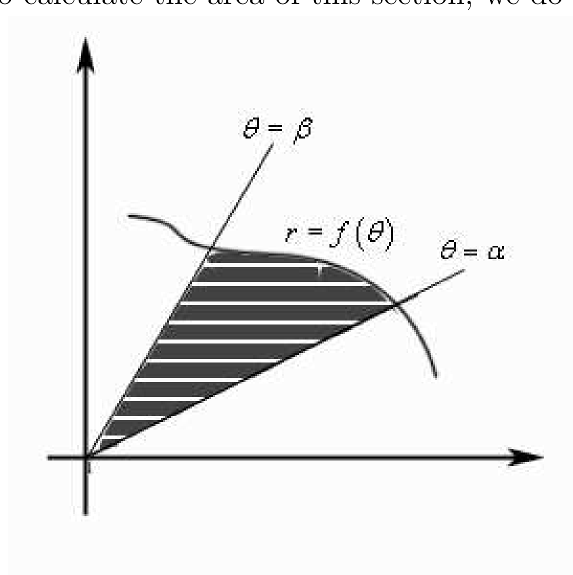
$$\frac{\pi r^2}{A} = \frac{2\pi}{\vartheta}.$$

Solving, we get

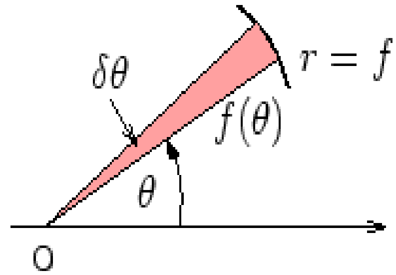
$$A = \frac{1}{2}r^2\vartheta.$$

□

We shall now work out how to find the area of a region given in polar coordinates. Suppose we are given a region like the one illustrated below (so a function of the form $r = f(\vartheta)$ where the radius depends upon the angle). To calculate the area of this section, we do the following:



(*i*) Break up the region into small segments with equal sized interior angles of size $\Delta\vartheta$ as illustrated below:



(*ii*) Observe that for small $\Delta\vartheta$, these segments are approximately segments of circles. This means the area can be approximated by the formula $r^2\vartheta/2$ where r is the radius. Observe however that at any point ϑ, the radius is $f(\vartheta)$, so the area can be approximated by

$$\frac{f(\vartheta)^2\Delta\vartheta}{2}.$$

(*iii*) Adding up all the segments, we can approximate the whole area by the sum

$$\sum_{i=1}^{n}\frac{f(\vartheta)^2\Delta\vartheta}{2}.$$

(*iv*) Taking smaller and smaller subdivisions, this sum gets closer to the actual answer. But this is a Riemann sum, so we get:

Result 1.2. The area of a region R bounded by a polar function between $\vartheta = \alpha$ and $\vartheta = \beta$ is given by the integral

$$\int_{\alpha}^{\beta}\frac{[f(\vartheta)]^2}{2}d\vartheta.$$

To illustrate, we look at a number of examples.

Example 1.3. Recall that calculating the area of a circle using Cartesian coordinates was very complicated - it involved a complicated trig substitution and then trig identities to simplify to something we can integrate. We shall show how easy it becomes using polar coordinates instead. If C is a circle of radius R, then its polar equation is $f(\vartheta) = R$ where $0 \leqslant \vartheta \leqslant 2\pi$. Thus its area will be

$$\int_0^{2\pi}\frac{R^2}{2}d\vartheta = \frac{R^2}{2}x\Big|_0^{2\pi} = \pi R^2.$$

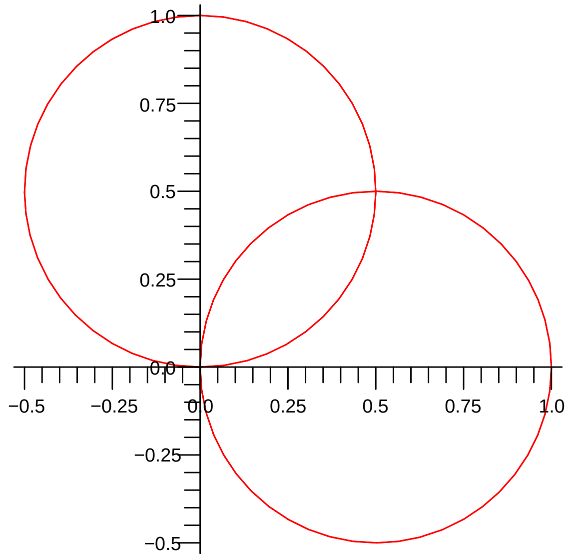
Example 1.4. Find the area of one of the petals in the eight leafed rose given by $f(\vartheta) = \sin(4\vartheta)$.
In order to calculate this, we first need to find out the limits on ϑ. One leaf occurs when r oscillates from 0 back to 0. The first time $r = 0$ is

when $\vartheta = 0$, and the second time is when $\vartheta = \pi/4$. Thus the area will be

$$\int_0^{\pi/4} \frac{\sin^2(4\vartheta)}{2} d\vartheta = \int_0^{\pi/4} \frac{1-\cos(8\vartheta)}{4} dx = \frac{1}{8}\left[\vartheta - \frac{\sin(8\vartheta)}{8}\right]_0^{\pi/4} = \frac{\pi}{16}.$$

Example 1.5. Find the area of the region bounded between $r = \sin(\vartheta)$ and $r = \cos(\vartheta)$.

Before we try to calculate an area, we sketch the graph to give us an idea of what to look for.



Observe that until the value $\vartheta = \pi/4$, the radius function is given by $f(\vartheta) = \sin(\vartheta)$. Then between $\pi/4$ and $\pi/2$, the radius function is given by $f(\vartheta = \cos(\vartheta)$. Thus the total area will be

$$\int_0^{\pi/4} \frac{\sin^2(\vartheta)}{2} d\vartheta + \int_{\pi/4}^{\pi/2} \frac{\cos^2(\vartheta)}{2} d\vartheta = \frac{1}{2}\left[\vartheta - \frac{\sin(2\vartheta)}{2}\right]_0^{\pi/4} + \frac{1}{2}\left[\vartheta + \frac{\sin(2\vartheta)}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] + \frac{1}{2}\left[\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\right] = \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] + \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{\pi}{4}.$$

2. Arc Length

We can also modify the arc length formula so we can find arclengths of polar functions. Specifically, if we assume that $r = f(\vartheta)$, then we have $x = f(\vartheta)\cos(\vartheta)$ and $y = f(\vartheta)\sin(\vartheta)$, so they can both be considered as parametric equations in ϑ. Observe that

$$dx/dt = f'(\vartheta)\cos(\vartheta) - f(\vartheta)\sin(\vartheta)$$

and

$$dy/dt = f'(\vartheta)\sin(\vartheta) + f(\vartheta)\cos(\vartheta).$$

Thus

$$(dx/dt)^2 + (dy/dt)^2 = (f'(\vartheta)\cos(\vartheta) - f(\vartheta)\sin(\vartheta))^2 + (f'(\vartheta)\sin(\vartheta) + f(\vartheta)\cos(\vartheta))^2$$

$$= (f'(\vartheta))^2 \cos^2(\vartheta) - 2f(\vartheta)f'(\vartheta)\cos(\vartheta)\sin(\vartheta) + (f(\vartheta))^2\sin^2(\vartheta)$$
$$+(f'(\vartheta))^2 \sin^2(\vartheta) + 2f(\vartheta)f'(\vartheta)\cos(\vartheta)\sin(\vartheta) + (f(\vartheta))^2\cos^2(\vartheta)$$
$$= (f(\vartheta))^2 + (f'(\vartheta))^2.$$

Then using the parametric arclength formula we get:

Result 2.1. The length of a curve with polar equation $r = f(\vartheta)$ with $\alpha \leqslant \vartheta \leqslant \beta$ is

$$\int_\alpha^\beta \sqrt{(f(\vartheta))^2 + (f'(\vartheta))^2}dt.$$

We illustrate with an example.

Example 2.2. Find the arclength of the region bounded between $r = \cos(\vartheta)$ and $r = \sin(\vartheta)$.
This is the graph we considered for the last example. We observe that it follows $f(\vartheta) = \sin(\vartheta)$ for $0 \leqslant \vartheta \leqslant \pi/4$ and $g(\vartheta) = \cos(\vartheta)$ for $\pi/4 \leqslant \vartheta \leqslant \pi/2$. We also observe that $f'(\vartheta) = \cos(\vartheta)$ and $g'(\vartheta) = -\sin(\vartheta)$, so

$$\int_0^{\pi/4} \sqrt{(f(\vartheta))^2 + (f'(\vartheta))^2}dt + \int_{\pi/4}^{\pi/2} \sqrt{(g(\vartheta))^2 + (g'(\vartheta))^2}dt$$
$$= \int_0^{\pi/4} 1 d\vartheta + \int_{\pi/4}^{\pi/2} 1 d\vartheta = \frac{\pi}{2}$$