Section 11.1: Sequences

In this section, we shall study something of which is conceptually simple mathematically, but has far reaching results in so many different areas of mathematics - sequences.

1. The Definition of a Sequence and Examples

We start with the definition of a sequence.

Definition 1.1. A sequence is an infinite succession of numbers

$$a_1, a_2, a_3, \ldots$$

written in a specificied order. The term a_1 is called the first term, the term a_2 is called the second term, and in general, the term a_n is called the nth term.

There are many different ways to represent and construct sequences.

- (i) We could write the nth term of a sequence as an expression in n (so a sequence can be realized as a function from the positive integers to the real numbers where the image of n is a_n).
- (ii) We could write an expression for a sequence as a function of prior terms (this is sometimes referred to as a "recursive" sequence).
- (iii) Some times there is no nice way to represent a sequence because there is no pattern which illustrates the terms of the sequence.

To illustrate, we look at some examples.

Example 1.2.
- (i) We define a sequence as $a_n = n$. This sequence will consist of the integers written in consecutive order $\{1, 2, 3, 4, \ldots\}$.
- (ii) We can define an interesting sequence as follows,

  $$a_n = n\text{th decimal place in the decimal expansion of } \pi.$$

  Observe that there is no nice way of writing down this sequence as a function or recursive sequence.
- (iii) We can define the recursive sequence $a_1 = 1$, $a_2 = 1$ and for $n > 2$ we define $a_n = a_{n-1} + a_{n-2}$. The first few terms are: $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$. This sequence is called the Fibonacci sequence and arises in many strange natural and physical situations.

Our main interest are sequences that can be described through the use of recursion or a formula - it is sequences like these which we can determine most information about. This means that the process of finding a formula is very important when examining sequences. We look at a couple of examples.

Example 1.3. Find formulas for the following sequences (either recursive or as a function of n).

(i)
$$\{1, 4, 9, 16, 25, 36, \ldots\}.$$
Observe that this sequence is the ascending list of squares of integers. Therefore, a formula will be $a_n = n^2$.

(ii)
$$\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\}.$$
In this case, the sequence is changing from positive to negative and the denominator is ascending power of 2. Therefore a formula will be
$$a_n = (-1)^{n+1}\frac{1}{2^n}.$$

(iii)
$$\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots\}.$$
The numerator is running over consecutive odd numbers and the denominator is running over consecutive even numbers. Thus we have
$$a_n = \frac{2n-1}{2n}.$$

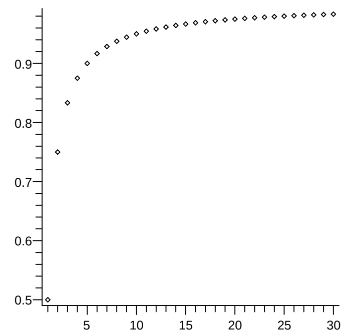
(iv)
$$\{1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1 \ldots\}.$$
This function does not look like it can be described nicely as a function of n. Therefore, we try to define it recursively. Notice that,
$$\begin{gathered}
a_1 = 1\\
a_2 = 0\\
a_3 = a_2 - a_1 = -1\\
a_4 = a_3 - a_2 = -1 - 0 = -1\\
a_5 = a_4 - a_3 = -1 - (-1) = 0\\
a_6 = a_5 - a_4 = 0 - (-1) = 1\\
a_7 = a_6 - a_5 = 1 - 0 = 1\\
a_8 = a_7 - a_6 = 1 - 1 = 0\\
a_9 = a_8 - a_7 = 0 - 1 = -1\\
a_{10} = a_9 - a_8 = -1 - 0 = -1\\
a_{11} = -1 - (-1) = 0
\end{gathered}$$
so the sequence will be $a_1 = 0$, $a_2 = 1$ and $a_n = a_{n-1} - a_{n-2}$ for $n > 2$.

Since a sequence is a function whose domain is the positive integers, we can construct the graph of $\{a_n\}$. Observe that the graph will **not** be continuous, but will be discrete (WHY?). We consider an example.

Example 1.4. Draw the graph of the sequence

$$a_n = \frac{2n-1}{2n}$$

and use it to guess what the limit of the sequence is as n gets large.



Observe that the values of the sequence are getting closer to 1 as n gets larger, so we would guess the limit to be 1.

2. Limits of Sequences

As we with the last example, given a sequence, we may want to describe the long term behavior of the sequence, or equivalently, the behavior of the sequence as we allow $n \to \infty$. We formalize this idea as follows:

Definition 2.1. A sequence $\{a_n\}$ has limit L and we write

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty$$

if we can make the terms of a_n as close to L as we like by taking n sufficiently large. If L exists, we say the sequence converges with limit L. Otherwise we say the limit diverges.

As we saw with the example, we could try to use the graph to determine the limit. Every sequence a_n is defined as some function of n, so every point of a_n lies on the graph of $f(x)$ where $f(n) = a_n$. This means that the function $f(x)$ associated to the sequence a_n will have the same limit, so we get the following:

Result 2.2. If $\lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.

Example 2.3. Find the limit of the sequence

$$a_n = \frac{(2n^2+1)(3n-1)}{5n^3-2}.$$

Observe that if

$$f(x) = \frac{(2x^2+1)(3x-1)}{5x^3-2}$$

then $f(n) = a_n$, so we can simply evaluate the limit

$$\lim_{x\to\infty} f(x) = \frac{6}{5}.$$

If the limit of a sequence does not exist, then it is said to diverge. However, there are many different types of divergence. One important type of divergence is when a sequence grows without bound (when the limit of the sequence is "∞"). We formalize:

Definition 2.4. We say the limit of the sequence a_n is ∞ and write $\lim_{n\to\infty} a_n = \infty$ if for every number M there is an N such that $a_n > M$ for all $n > N$.

Example 2.5. Show that the sequence $a_n = n$ is ∞ using the definition.

We need to show that for any value M, we can find an N such that for all $n \geqslant N$, $a_n \geqslant M$. But we can just choose N to be the integer closet, but larger, than M. Then $a_n = n \geqslant N \geqslant M$.

Limits have already been studied extensively (remember Calc 1), so it would be more efficient to try to use what we already know than develop lots of new rules. As we observed above, a sequence agrees with some function $f(x)$ at the integer points, so to find the limit of a sequence, we can use all the rules of limits we know for functions.

Result 2.6. (See Page 705 for details) If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is constant, then

(i) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
(ii) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
(iii) $\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n$
(iv) $\lim_{n\to\infty}(a_nb_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$
(v) $\lim_{n\to\infty}(a_n/b_n) = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$
(vi) $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p$

We also have the squeeze theorem:

Result 2.7. If $a_n \leqslant b_n \leqslant c_n$ and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

We also have the following intuitive result:

Result 2.8. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

To illustrate how to use these results, we shall look at some examples.

Example 2.9. Determine which of the following sequences converge and then find the limits if they exist.

(*i*) $a_n = n(n-1)$

The associated function to this sequence is $f(x) = x(x-1)$ which grows without bound as $x \to \infty$. Thus the sequence a_n diverges to ∞.

(*ii*)

$$\frac{(-1)^{n-1}n}{n^2+1}$$

Observe that

$$\left|\frac{(-1)^{n-1}n}{n^2+1}\right| = \frac{n}{n^2+1} \to 0$$

so it follows that $a_n \to 0$.

(*iii*) $a_n = \cos(n)$

Observe that the function $f(x) = \cos(x)$ oscillates between -1 and 1, so it seems reasonable that $\cos(x)$ should not converge. To see it does not converge, we note that we can always find values "close" to 1 and "close" to 0. To get numbers "close" to 1, we can take the integer values of $10^k\pi$ by dropping off all the decimals. As we take larger and larger values of k, this will approximate $10^k\pi$ better and better, so the value will get closer to 1. To get a number close to 0, we can take the integer values of $(10^k-1)(\pi/2)$ by dropping off all decimal places. As we take larger and larger values of k, this will approximate $(10^k-1)\pi/2$ better and better, so the value will get closer to 0.

(*iv*) $\cos(2\pi x)$

In this case, the fact that the corresponding function oscillates seems to suggest that this function should diverge as the last function did. Observe however that $f(n) = \cos(2n\pi) = 1$ for all integers n. In particular, it converges to 1. This shows that similar sequences can have very different divergence properties.

(*v*) $a_n = \cos(1/n)$

Observe that as $n \to \infty$, $1/n \to 0$, so it follows that $\lim_{n\to\infty}\cos(1/n) = \cos(0) = 1$.

(*vi*) $a_n = n^2e^{-n}$

Observe in this case the corresponding function is

$$f(x) = \frac{n^2}{e^x}.$$

Using L'hopitals rule (type ∞/∞), we get

$$\lim_{x\to\infty}\frac{n^2}{e^x} = \lim_{x\to\infty}\frac{2n}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

so $\lim_{n\to\infty} a_n = 0$.

(*vii*)

$$\frac{\ln(n)}{\ln(2n)}$$

In this case,

$$\lim_{x\to\infty}\frac{\ln(x)}{\ln(2x)}=\lim_{x\to\infty}\frac{\ln(x)}{\ln(2)+\ln(x)}=\lim_{x\to\infty}\frac{1}{\frac{\ln(2)}{\ln(x)}+1}=1$$

so $\lim_{n\to\infty} a_n = 1$.

Example 2.10. For what values of r does the sequence $a_n = r^n$ converge?
Observe that if $r = 1$ then the sequence $a_n = 1^n$ converges to 1. If $a_n = (-1)^n$, then it oscillates between -1 and 1 so will never converge. If $|r| > 1$, then $|r^n| \to \infty$, so the sequence diverges. If $|r| < 1$, then $|r^n| \to 0$¡ so the sequence converges. Thus we get $a_n = r^n$ converges if and only if $-1 < r \leqslant 1$.

There are certain types of sequences for which it is fairly easy to determine convergence or divergence.

Definition 2.11. A sequence a_n is called increasing if $a_n < a_{n+1}$ for all $n \geqslant 1$ and decreasing if $a_n < a_{n+1}$ for all $n \geqslant 1$. It is called monotonic if it is either increasing or decreasing.

Determining whether a monotonic sequence converges is fairly easy. To do this, we need the following definition.

Definition 2.12. A sequence a_n is bounded above if there is a number M such that $a_n \leqslant M$ for all $n \geqslant 1$. It is bounded below if $a_n \geqslant M$ for all $n \geqslant 1$. If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Observe that if a sequence is monotonic increasing and it converges, there must be a largest possible value (that it, it must be bounded above). Alternatively, if a_n is monotonic increasing but is not bounded from above, then the values will keep getting larger, so the sequence will not converge. Similar observations can be made about monotonic decreasing function, so we obtain the following nice result.

Result 2.13. Every bounded monotonic sequence converges.

Observe that it must be monotonic (since $a_n = \cos(n)$ is bounded but not convergent, but it is not monotonic). We finish with an example.

Example 2.14. Is the sequence $a_n = n + 1/n$ monotonic? Does it converge?
It is monotonic because if $f(x) = x + 1/x$, then $f'(x) = 1 - 1/x^2 > 0$ for $x > 0$ (so it is an increasing function, and so the sequence a_n is increasing). It diverges because $a_n + 1/n > n$ which diverges.