Section 11.10: Taylor and Maclaurin Series

1. Taylor and Maclaurin Series Definitions

In this section, we consider a way to represent **all** functions which are "sufficiently nice" around some point. By "sufficiently nice", we mean that every possible derivative of $f(x)$ exists. Rather than referring to it as such, we use the following definition:

Definition 1.1. We call a function C^n if it is differentiable at least n-times (so the $n$the derivative exists). We call it C^∞ if every possible derivative exists.

In Calculus 1, we used linear approximation to approximate the value of a function at a point. This is a very good approximation if the graph does not admit too much curvature, but it can cause problems if the graph curves a lot. Therefore, instead of approximating by a linear function, we can approximate by a polynomial (the next easiest type of function to do Calculus on). Of course, the higher the degree of the polynomial we are approximating by, the closer we expect the approximation to be, so the best approximation of a function by a polynomial is when it can be represented as a power series. Therefore, suppose that $f(x)$ can be represented by a power series:

$$f(x) = \sum c_n (x-a)^n.$$

Question 1.2. What do the coefficients of this power series tell us about $f(x)$?

- First observe that $f(a) = c_0$, so the constant term and the value of f at $x = a$ agree.
- Next,
$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$
so $f'(a) = c_1$ so the coefficient of the linear term and the first derivative of f at $x = a$ agree.
- Next,
$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$
so $f'(a) = c_1$ so the coefficient of the linear term and the first derivative of f at $x = a$ agree.
- Next,

$$f''(x) = 2c_2 + 3*2*c_3(x-a) + 4*3*c_4(x-a)^2 + 5*4*c_5(x-a)^3 + \dots$$

so $f''(a) = 2c_2$ so the second derivative of f at $x = a$ is 2! times the coefficient of the quadratic term.

- Next,

$$f'''(x) = 3*2*c_3 + 4*3*2c_4(x-a) + 5*4*3c_5(x-a)^2 + \dots$$

so $f'''(a) = 3*2c_3$ so the third derivative of f at $x = a$ is 3! times the coefficient of the cubic term.

- Next,

$$f^{(4)}(x) = 4*3*2*c_4 + 5*4*3*2c_5(x-a)^2 + \dots$$

so $f^{(4)}(a) = 4*3*2c_4$ so the fourth derivative of f at $x = a$ is 4! times the coefficient of the quadratic term.

Continuing in this fashion, we get that if $f(x)$ can be represented as a power series, then the nth derivative of the function $f(x)$ will be $n!$ times the coefficient of the nth term in the power series. Alternatively, if $f(x)$ can be represented as a power series around $x = a$, the nth coefficient will be equal to the nth derivative of $f(x)$ at $x = a$ divided by $n!$. We summarize.

Result 1.3. If f has a power series representation at $x = a$, so

$$f(x) = \sum c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(x)}{n!}.$$

This means that if a function can be represented by a power series at $x = a$, then it has the form:

$$\sum \frac{f^n(a)}{n!}(x-a)^n.$$

This motivates the following definitions.

Definition 1.4. If $f(x)$ is a C^∞ function, we call the power series

$$\sum \frac{f^n(a)}{n!}(x-a)^n$$

the Taylor series for $f(x)$ around $x = a$. In the special case when $a = 0$, we call the Taylor series

$$\sum \frac{f^n(a)}{n!}x^n$$

the Maclaurin series.

In general, given a C^∞ function $f(x)$, we can always construct its Taylor series around $x = a$. This lead to the natural question of when a Taylor series of a function agrees with the function. This is a fairly complicated problem - there are functions for which the Taylor series only agrees with the function at the point $x = a$. A complete answer to this question is beyond this course - we shall just accept the following answer:

Result 1.5. Suppose

$$T_N(x) = \sum_{n=0}^{N} \frac{f^n(a)}{n!}(x-a)^n$$

is the Nth Taylor polynomial of $f(x)$ around $x = a$ and

$$R_N(x) = \sum_{N+1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

is the rest of the sum. If $\lim_{N\to\infty} R_N(x) = 0$ for $|x-a| < R$, then f is equal to its Taylor series on the interval $|x-a| < R$.

This result basically says that provided the latter part of the Taylor sum goes to 0, then the Taylor series of a function is equal to the function on its interval of convergence. Most of the Taylor series we shall be considering will be equal to the corresponding functions. We shall look at the classic functions where the Taylor series is equal to the function on its whole interval of convergence.

Example 1.6. Find the Maclaurin Series of the following functions:

(i) $f(x) = e^x$

We need to find an expression for the nth derivative. Observe that $f(0) = 1$, $f'(x) = e^x$, so $f'(0) = 1$. Likewise, for any integer n, we have $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = 1$. This means the Taylor series for $f(x) = e^x$ will be

$$\sum \frac{x^n}{n!}.$$

This power series converges for all x, so we have

$$e^x = \sum \frac{x^n}{n!}$$

for all values of x.

(ii)

$$f(x) = \frac{1}{1-x}$$

Observe that this is the sum formula for a geometric series, so the Taylor series around $x = 0$ will be

$$\sum x^n.$$

This power series converges for x in $(-1, 1)$, so we have

$$\frac{1}{1-x} = \sum x^n$$

provided $|x| < 1$.

(*iii*) $f(x) = \sin(x)$ In this case we have $f(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(0) = -\sin(0) = 0$, $f'''(0) = -\cos(0) = -1$, $f^{(4)}(0) = \sin(0) = 0$, and the series keeps repeating after this. We observe the following:

- every even term is zero
- every other odd term is negative

This suggests that the series will be

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

This power series converges for all x, so we have

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

for all values of x.

(*iv*) $f(x) = \cos(x)$

In this case we have $f(0) = 1$, $f'(0) = -\sin(0) = 0$, $f''(0) = -\cos(0) = -1$, $f'''(0) = -\sin(0) = 0$, $f^{(4)}(0) = \cos(0) = 1$, and the series keeps repeating after this. We observe the following:

- every odd term is zero
- every other even term is negative

This suggests that the series will be

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

This power series converges for all x, so we have

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

for all values of x.

(*v*) $f(x) = \arctan(x)$

We know that

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

so

$$\arctan(x) = \sum (-1)^n \frac{x^{2n+1}}{2n+1}.$$

This power series converges for x in $(-1, 1]$, so we have

$$\arctan(x) = \sum (-1)^n \frac{x^{2n+1}}{2n+1}$$

provided $|x| < 1$ or $x = 1$.

2. Applications of Taylor and Maclaurin Series

Taylor series have many applications, but one important one we can understand is that of approximating irrational numbers. We illustrate with some examples.

Example 2.1. (*i*) Approximate the value of e using Taylor series.

We observe that

$$e^x = \sum \frac{x^n}{n!}$$

for all x, so in particular

$$e = e^1 = \sum \frac{1}{n!}.$$

Therefore, the value for e can be approximated by taking partial sums of this series. Obviously, the more terms we take in the partial sum, the closer we will get to the values. Taking $N = 10$, we get

$$e \sim 1/1! + 1/2! + \cdots + 1/10! \sim 2.71828$$

and on the calculator we get $e \sim 2.718282$, so the 10th partial sum is very close.

(*ii*) Approximate the value of π using Taylor series.

We observe that $\arctan(1) = \pi/4$, or $\pi = 4\arctan(1)$. Also, on the interval $(-1, 1]$, we have

$$\arctan(x) = \sum (-1)^n \frac{x^{2n+1}}{2n+1},$$

so it follows that π can be approximated by taking partial sums of

$$4 \sum (-1)^n \frac{1}{2n+1}.$$

Obviously, the more terms we take in the partial sum, the closer we will get to the values. Taking $N = 100$, we get

$$\pi \sim 4(1 - 1/3 + 1/5 \cdots - 1/19) \sim 4 * (0.787873) = 3.151492$$

which is within 1 decimal place. Taking $N = 1000$, we get

$$\pi \sim 4(1 - 1/3 + 1/5 \cdots - 1/19) \sim 4 * (0.785648) = 3.142592$$

which is within 2 decimal places of the actual answer. In general, to estimate the value of π, we can take larger and larger partial sums (this is one of the ways they do this!!!!)

Taylor series can also be used to study functions which we know exist but do not have a way of writing them down.

Example 2.2. Recall that the function

$$f(x) = \frac{\sin(x)}{x}$$

does not have an algebraic antiderivative (try as you may, the TI89 will not give you an answer). However, we can use Taylor series to come up with a power series representation of its antiderivative. Since

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

for all values of x, we will have

$$\frac{\sin(x)}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

provided $x \neq 0$. Thus

$$\int \frac{\sin(x)}{x} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}.$$

Observe that in order to multiply the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

by $1/x$ we simply multiplied each term by $1/x$. In general, provided two power series converge, we can perform all the standard algebraic operations on them such as multiplication, composition, etc. This allows us to construct power series for many other functions than the standard few we have derived.

Example 2.3. Find the first few terms for a power series representation for the following functions:

(i)

$$f(x) = x^2 e^{-x}$$

We have

$$e^x = \sum \frac{x^n}{n!}$$

so

$$e^{-x} = \sum (-1)^n \frac{x^n}{n!}.$$

This means that

$$x^2 e^{-x} = x^2 \sum (-1)^n \frac{x^n}{n!} = \sum (-1)^n \frac{x^{n+2}}{n!}.$$

(ii) $e^x \sin(x)$ Here we have

$$\begin{aligned} e^x \sin(x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3!} + \dots\right) \\ &= x - \frac{x^3}{6} + x(x - \frac{x^3}{6}) + \frac{x^2}{2}(x - \frac{x^3}{6}) + \dots \\ &= x + x^2 + \frac{x^3}{3} + \dots \end{aligned}$$

For fun, we look at one more example of explicitly finding a Taylor series.

Example 2.4. Find the Taylor series of

$$f(x) = \frac{1}{x}$$

about $x = 1$.

First we must find a general term for the nth derivative. Observe that $f(1) = 1$, $f'(x) = -\frac{1}{x^2}$, so $f'(1) = -1$, $f''(x) = \frac{2}{x^3}$, so $f''(1) = 2$, $f'''(x) = -\frac{2*3}{x^4}$, so $f'''(1) = -2*3$, $f^{(4)}(x) = \frac{2*3*4}{x^5}$, so $f^{(4)}(1) = 2*3*4$. In general, it looks like the nth derivative will have the form

$$f^n(1) = (-1)^n n!$$

This means the Taylor series for $f(x) = 1/x$ around $x = 1$ will be

$$\sum (-1)^n (x-1)^n.$$

Observe that we could have obtained this formula by substituting $1-x$ in for x in the expansion for $1/(1-x)$.