Section 11.2: Series

In this section, we shall examine one of the most important things we can do with sequences - adding up their the terms. Amongst other things, Riemann sums are an example of some series - in a Riemann sum, we add up "infinitely many" strips of area to estimate area under a graph. We shall be looking at much more general series.

1. The Definition of a Series and Convergence

We start with a definition.

Definition 1.1. If a_n is a sequence, we can add up all the terms $a_1 + a_2 + \dots$. We call this sum an infinite series (or series for short) and we denote it by

$$\sum_{i=1}^{\infty} a_n$$

or

$$\sum a_n.$$

The natural question to ask is what the value of this sum is. The problem of course is that this sum is of an infinite number of things, so it seems counterintuitive that it should even have a sum which is not infinite. Of course, we already know that there are lots of infinite sums which do have finite sums (take a Riemann sum for example). So we need to determine a way to show whether or not an infinite sum sums to a finite number and then how to find out what this finite number is. For this, we do the following:

(i) Suppose a_n is a sequence. Denote by S_N the sum of the first N terms. So

$$S_N = \sum_{i=1}^{N} a_n.$$

We call S_N the Nth partial sum of the series

$$\sum a_n.$$

(ii) Observe that the set of partial sums forms a sequence: that is S_1, S_2, S_3, $\dots$, form a sequence of numbers (every sum is finite, so they can be calculated).

(iii) If the infinite sum

$$\sum a_n$$

exists and is equal to L, then

$$\lim_{N\to\infty} S_N = L$$

(the partial sums must get closer to L as N gets larger). Likewise, if

$$\lim_{N\to\infty} S_N = L,$$

it means that the more terms we add, the closer we get to L.

(*iv*) With this in mind, we say that a series

$$\sum a_n$$

converges to L if the sequence of partial sum S_N converges to L, so

$$\lim_{N\to\infty} S_N = L.$$

Under these circumstances, we say that the series is convergent and we write

$$\sum a_n = a_1 + \ldots a_n + \cdots = L$$

and say that L is the sum of this series. If the series does not converge, we say it diverges.

We look at a couple of easy examples.

Example 1.2. (*i*) Find S_5 S_6 and $S_6 - S_5$ of the series

$$\sum \frac{n}{n+1}.$$

$$S_5 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

$$S_6 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}$$

$$S_6 - S_5 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}\right) = \frac{6}{7}$$

(*ii*) Does the series

$$\sum \frac{n}{n+1}$$

seem to converge? Why or why not?

Observe that

$$\lim_{N\to\infty} S_{N+1} - S_N = \lim_{N\to\infty} \frac{N+1}{N+2} = 1.$$

This means that after a certain point, we shall keep adding numbers to our series very close to 1. This suggests that the series cannot possibly converge.

2. Basic Rules of Convergence

To determine whether or not a series converges, we need to consider the corresponding sequence of partial sums (so this requires a **thorough** knowledge of sequences). As the previous example suggests, there are some fairly obvious restrictions which must be true of convergent sequences.

Result 2.1. Suppose

$$\sum a_n$$

converges. Then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Suppose that $\sum a_n$ converges to L. Observe that

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty} S_N - S_{N-1}.$$

This means

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = L - L = 0.$$

□

In particular, this result implies the following important test **for divergence**.

Result 2.2. If

$$\lim_{n\to\infty} a_n \neq 0$$

or does not exist, then the series

$$\sum a_n$$

diverges.

Since series are determined using a sequence of partial sums, its seems that the limit laws we developed for sequences should also hold. These laws are:

Result 2.3. If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then the series $\sum ca_n$, $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ are also convergent and

(*i*)

$$\sum ca_n = c\sum a_n$$

(*ii*)

$$\sum(a_n + b_n) = \sum a_n + \sum b_n$$

(*iii*)

$$\sum(a_n - b_n) = \sum a_n - \sum b_n$$

We need to be careful with this example - it requires that a_n and b_n are convergent. For example, we could not take $\sum 1$ and $\sum 1$, which are both divergent series and find

$$\sum(1-1) = \sum 0 = 0 = \sum 1 - \sum 1$$

because the left hand side does not make sense (since neither of the series diverge).

3. Examples

To illustrate our observations, we look at some examples.

Example 3.1. Determine, with reasons, which of the following are convergent series. When convergent, find the sum if possible.

(*i*)

$$\sum \frac{1}{n}$$

(This is a special series called the harmonic series).

Observe that $\lim a_n = 0$, so we cannot immediately conclude that this sum diverges. Instead we shall look at the partial sums to see if we can see a pattern. Observe that

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$$

$$S_8 = 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}) > 1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}) = 2\frac{1}{2}$$

If we continue this process, we see that if we take the sum S_{2^n} we get

$$S_{2^n} > 1 + n\frac{1}{2}$$

for every possible n. This means that the sum grows without bound and hence the sum cannot converge.

(*ii*)

$$\sum \frac{2}{n^2-1}$$

(This is a special series called a telescopic series).

Observe that

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum(\frac{1}{n-1} - \frac{1}{n+1}).$$

If we evaluate the Nth partial sum, we get

$$(1-\frac{1}{3})+(\frac{1}{2}-\frac{1}{4})+(\frac{1}{3}-\frac{1}{5})+(\frac{1}{4}-\frac{1}{6})+\cdots+(\frac{1}{N-2}-\frac{1}{N})+(\frac{1}{N-1}-\frac{1}{N+1})$$

$$= 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N-1}$$

Letting $N \to \infty$, we get

$$\lim_{N\to\infty} S_N = 1 + \frac{1}{2}$$

so the series converges to $\frac{3}{2}$.

(iii)

$$\sum \arctan(n)$$

Observe in this case that $\lim_{n\to\infty} \arctan(n) = \pi/2$, so the series cannot possibly converge.

(iv)

$$\sum \ln\left(\frac{n}{2n+5}\right)$$

Observe that

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{2+5/n}\right) = \ln(\frac{1}{2}) \neq 0$$

so this series cannot possibly converge.

(v)

$$\sum \frac{2}{n^2+4n+3}$$

This is also a telescopic series, observe that

$$\sum \frac{2}{n^2+4n+3} = \sum \frac{1}{n+1} - \frac{1}{n+3}.$$

As with the other example, when the partial sums are calculated, the result will just be the first two terms of the sequence

$$\frac{1}{n+1},$$

so the series converges to

$$\frac{1}{2} + \frac{1}{3}.$$

Example 3.2. Suppose that r and a are numbers. The series

$$\sum ar^n$$

is called a geometric series. and determining whether it converges and when it does, what it converges to, is fairly straight forward.
First suppose that $|r| \geqslant 1$. Then $\lim_{n\to\infty} ar^n \neq 0$ so it follows that the sum

$$\sum ar^n$$

does not converge. Therefore, the only values of r for which it may converge are when $|r| < 1$. Suppose that $|r| < 1$. Then if S_N is the $N$the partial sum, observe that

$$(1-r)S_N = (a + ar + \ldots ar^N) - (ar + ar^2 + \ldots ar^{N+1}) = a - ar^{N+1}.$$

This means that the Nth partial sum can be calculated using the formula

$$S_N = \frac{a(1-r^N)}{(1-r)}.$$

To find the sum, we need to take

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{(1-r)} = \frac{a}{1-r}$$

since $|r| < 1$. Thus we get the following result:

Result 3.3. If $\sum ar^n$ is a geometric series, it converges if and only if $|r| < 1$. If it converges then it converges to

$$\frac{a}{1-r}.$$

This result makes finding certain sums very easy.

Example 3.4. Does $1 + .4 + .16 + .064 + \ldots$ converge, and if so, what does it converge to?
This is a geometric series with first term $a = 1$ and common ratio $.4$, so it converges with sum $1/(1-0.4) = 1/(0.6)$.

Geometric series are very useful. They can help transform a decimal into a fraction.

Example 3.5. Write the decimal $0.7\bar{2}8$ as a fraction.
We observe that

$$0.7\bar{2}8 = \frac{728}{1000} + 728 * \frac{1}{(1000)^2} + 728 * \frac{1}{(1000)^3} + 728\frac{1}{(1000)^4} + \ldots$$

This is a geometric series with first term $a = 728/1000$ and common ratio $1/1000$. Therefore, we can use the geometric sum formula to get

$$\begin{aligned} 0.7\bar{2}8 &= \frac{728}{1000} + 728 * \frac{1}{(1000)^2} + 728 * \frac{1}{(1000)^3} + 728\frac{1}{(1000)^4} + \ldots \\ &= \frac{\frac{728}{1000}}{1-\frac{1}{1000}} = \frac{\frac{728}{1000}}{\frac{999}{1000}} = \frac{728}{999} \end{aligned}$$