Section 11.3: The Integral Test

Most of the series we have looked at have either diverged or have converged and we have been able to find what they converge to. In general however, the problem is much more difficult than what we have seen. Convergence can be broken up into two problems 1) Does a series converge? and 2) What does it converge to? To tackle the first problem, we shall develop a whole new toolbox based on results we have already used in Calculus. The second problem is much more difficult and requires more advanced mathematics than we currently have. Therefore, our focus for most of this chapter will be on the first problem (though we shall occasionally use the second).

1. Using Integrals to Determine Convergence - an Example

We start with an example.

Example 1.1. Show that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Consider the function $f(x) = 1/x^2$ which coincides with this sum for integer values. We can approximate the area under this curve using a Riemann sum. If we take the right hand Riemann sum with subdivisions of length 1, then we get the following infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is the same as the series we are trying to find. If we ignore the first term of the sum and just consider the infinite sum

$$\sum_{n=2}^{\infty}$$

then the value of this sum will be strictly bounded by the area under the graph from 1 to ∞ (since the right hand sum produces an underestimate). But this area can be estimated using an improper integral. Summarizing, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \leqslant 1 + \int_{1}^{\infty} \frac{1}{x^2} dx.$$

In particular, the sum will only converge if the integral converges. Calculating, we have

$$\int_1^\infty \frac{1}{x^2}dx = -\frac{1}{x}\bigg|_1^\infty = 1.$$

Therefore, since the interval converges and is larger than the sum, the sum must converge. Hence the sum

$$\sum_{n=1}^\infty \frac{1}{n^2}$$

converges.

WARNING: Just because it converges does not mean we have found the actual value of the sum. In fact, all we have shown is that the sum

$$\sum_{n=1}^\infty \frac{1}{n^2} = 1 + \sum_{n=2}^\infty \frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2}dx = 2.$$

2. The Integral Test

The method we used in our example approximating the sum with an integral can be applied to lots of other cases to show that a series converges (observe that we did not find what the sum was (it is in fact $\pi^2/6$)). There are just a couple of things which are necessary for it to work:

- (i) The sequence in the series must be positive for all values past a certain point.
- (ii) The sequence in the series must be decreasing after a certain point (to guarantee that we will be getting an underestimate).
- (iii) The function corresponding to the sequence is continuous (so we can take the Riemann sum).

If these things are satisfied, then we can apply the following:

Result 2.1. (The Integral Test) Let

$$\sum a_n$$

be a series with positive terms and let $f(x)$ be the function that results when n is replaced by x. If f is decreasing and continuous on $[a, \infty)$ then

$$\sum a_n$$

and

$$\int_a^\infty f(x)$$

either both converge or diverge.

Observe that we only start at a - this was illustrated in the example because if we tried to integrate from 0 to ∞, the integral will have diverged. Of course, the key point is that the first few terms will not affect divergence or convergence - it is the ultimate behavior which counts and this is measured by the integral. We illustrate the power of the integral test with a few examples.

Example 2.2. Show that the harmonic series diverges.
Observe that the harmonic series

$$\sum \frac{1}{n}$$

agrees with the function $f(x) = 1/x$. However,

$$\int_1^\infty \frac{1}{x} dx = \ln(x)\Big|_1^\infty$$

which diverges. Since the integral diverges, the corresponding series must diverge.

Example 2.3. Show that the sum

$$\sum \frac{1}{n^3 + n}$$

converges.
First observe that

$$\frac{1}{n^3 + n} \leqslant \frac{1}{n^3}$$

since $n \geqslant 1$. Therefore

$$\sum \frac{1}{n^3} \geqslant \sum \frac{1}{n^3 + n}.$$

Using the integral test, we have

$$\int_1^\infty \frac{1}{x^3} dx = -\frac{1}{2x^2}\Big|_1^\infty = \frac{1}{2},$$

so

$$\sum \frac{1}{n^3}$$

converges, and hence so does

$$\sum \frac{1}{n^3 + n}.$$

Example 2.4. We define a p series to be the series

$$\sum \frac{1}{n^p}$$

where p is some integer. Find all values of p for which this series converges.
Clearly if $p \leqslant 0$, then this series diverges (because we shall be adding infinitely many numbers greater or equal to 1 together. Also, the functions will not be decreasing for $p < 0$, so we could not apply the integral

test anyway! Also, the case $p = 1$ was considered above (it is the harmonic series). Therefore, we may assume that $p > 0$ and $p \neq 1$. The corresponding integral will be

$$\int_1^\infty \frac{1}{x^p} dx = \int_1^\infty x^{-p} dx = \lim_{k\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^k = \lim_{k\to\infty} \frac{k^{1-p}}{1-p} - \frac{1}{1-p}$$

The only part of this limit which could diverge is the term k^{1-p}, and this will diverge if and only if $1 - p > 0$ or $p < 1$. Therefore, a p-series

$$\sum \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example 2.5. Determine which values of p the series

$$\sum e^{-np}$$

converges.

If $p \leqslant 0$, this clearly diverges, so we assume $p < 0$. This is a decreasing and continuous function, so we apply the integral test:

$$\int_1^\infty e^{-xp} dx = \lim_{k\to\infty} -\frac{e^{-xp}}{p}\bigg|_1^k = \lim_{k\to\infty} -\frac{e^{-kp}}{p} + \frac{e^{-p}}{p} = \frac{e^{-p}}{p}$$

This integral converges for all $p > 0$, so the series converges for all $p > 0$.

3. Estimating the Sum of a Series

Though out main concern is determining whether a series converges, we shall briefly discuss how we can approximate the value of a series which does converge using the integral test. If a series converges, then the more terms we add up, the closer the value will be to the actual value of the sum. Suppose that the actual value of a series is S and the nth partial sum is denoted by S_n. Then we define the remainder

$$R_n = S - S_n,$$

the difference between the actual value of the series and the partial sum. Obviously, the smaller the remainder is, the closer the partial sum is to the actual value of the series. We can use integrals to determine the remainder:

Result 3.1. Suppose $f(n) = a_n$ where f is continuous, positive, decreasing for $x \geqslant n$ and

$$\sum a_n$$

is convergent. If $R_n = S - S_n$, then

$$\int_{n+1}^\infty f(x)dx \leqslant R_n \leqslant \int_n^\infty f(x)dx.$$

This result basically tells us that the remainder can be approximated by these two integrals. We illustrate.

Example 3.2. Find the value of n which guarantees that the remainder will be within 0.1 of the actual value for the sum

$$\sum \frac{1}{n^2}.$$

Observe that

$$\int_n^\infty \frac{1}{x^2} dx = -\frac{1}{x}\Big|_n^\infty = \frac{1}{n}$$

If we choose $n = 10$, then we have

$$\frac{1}{11} \leqslant R_{10} \leqslant \frac{1}{10}$$

so only 10 terms in the sum are needed to get the value of

$$\sum \frac{1}{n^2}$$

within 0.1 of the actual answer. Using the TI89 sum option, we have $R_{10} = 1.54977$. Observe that $\pi^2/6 = 1.64493$, so it is within 0.1 of the actual answer!