Section 11.4: The Comparison Tests

1. The Comparison Test

In this section, we shall consider new ways to test for convergence of series. The first thing we shall consider is comparison of series - if we know that a particular series converges, then other series which behave in the same way will also probably converge. Likewise, if a given series diverges, then series which behave in the same way will also probably diverge. This suggests the following test.

Result 1.1. (The Comparison Test) Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all but finitely many values of n, then $\sum a_n$ is convergent.
(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all but finitely many values of n, then $\sum a_n$ is divergent.

Proof. First observe that if the equalities stated hold for all but finitely many n, then after some point in the sequence N, the equality must always hold. Since the convergence or divergence of a series is independent of the first few terms, we can ignore these terms and just consider the series starting from N where the equalities hold. Therefore without loss of generality, we shall assume that the equalities hold for all values of n.
If $\sum b_n$ converges, let $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} b_i$ and $t = \sum_{i=1}^{\infty} b_n$. Since both sequences are positive, s_n and t_n are both increasing, so both sequences are monotonic. Also, for every value of n, we have $t_n \geqslant s_n$. Since $t_n \to \infty$ as $n \to \infty$ we must have $t \geqslant t_n$ for all n, and therefore $s_n \leqslant t$ for all n. It follows that s_n is monotonic and bounded and so must converge.
If $\sum b_n$ diverges, then the values of $t_n \to \infty$ as $n \to \infty$. However, $s_n \geqslant t_n$ for all n, so we must have $s_n \to \infty$.

□

Warning. We must have the terms of the series positive for this test to be applied. For example, we could take $\sum 1/n^2$ and $\sum(-n)$. Clearly $-n < 1/n^2$ for all n, but we cannot conclude that $\sum(-n)$ converges because the terms are not positive.

We illustrate the comparison test with a number of examples.

Example 1.2. Which of the following converge/diverge?

(i)

$$\sum \frac{n-1}{n^3+3}$$

This series seems to behave like $1/n^2$, so we guess that it converges. To check, observe that

$$\frac{1}{n^2} \geqslant \frac{n-1}{n^3+3}$$

for all n, so the comparison test implies that it converges.

(*ii*)

$$\sum \frac{6n^2+1}{2n^3-1}$$

This series seems to behave like $1/n$, so we guess that it diverges. To check, observe that

$$\frac{1}{n} \leqslant \frac{6n^2+1}{2n^3-1}$$

for all n, so the comparison test implies that it diverges.

2. The Limit Comparison Test

Of course, the comparison test is not always helpful. For example, if we consider the series $\sum 1/(n+1)$, we would like to compare it to $\sum 1/n$ to conclude that it converges, but $1/n$ is always greater than $1/(n+1)$. However, we know for sure that it converges since it is just a shift of $1/n$. For series like these where the comparison test is not quite sufficient, we have the following test:

Result 2.1. (Limit Comparison Test) Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} a_n/b_n = c$ where c is a finite number and $c > 0$, then either both series converge or both series diverge.

We illustrate how to use this test with some examples.

Example 2.2. (*i*)

$$\sum \frac{1}{9n+6}$$

This series behaves like $1/n$, so we apply the limit comparison test. Evaluating, we get

$$\lim_{n\to\infty} \frac{1/n}{1/(9n+6)} = \frac{9n+6}{n} = 9 + \frac{6}{n} = 9.$$

Since $1/n$ diverges, it follows that $\sum 1/(9n+6)$ diverges too.

(*ii*)

$$\sum \frac{1}{(8n^2-3n)^{1/3}}$$

This series seems to behave like $1/n^(2/3)$, so we use the limit comparison test. Evaluating, we get

$$\lim_{n\to\infty} \frac{1/n^{2/3}}{1/(8n^2-3n)^{1/3}} = \left(\frac{8n^2-3n}{n^2}\right)^{1/3} = \left(8 - \frac{3}{n}\right)^{1/3} \to 2.$$

Since $1/n^{2/3}$ diverges, it follows that $\sum 1/(8n^2-3n)^{1/3}$ diverges too.

(iii)

$$\sum \frac{n-1}{n4^n}$$

This series seems to behave like $1/4^n$ so we shall test with the limit comparison.

$$\lim_{n\to\infty} \frac{1/4^n}{(n-1)/(n4^n)} = \frac{n4^n}{4^n(n-1)} = \frac{n}{(n-1)} = 1.$$

Since $1/4^n$ converges, it follows that $\sum(n-1)/n4^n$ converges too.