Section 11.6: Absolute Convergence and Ratio Tests

The tests we have so far developed are still not sufficient to determine convergence of series. This means we need to develop some more tests. The next two tests are two of the most important because they do not rely on results regarding other tests (like the comparison tests) and do not require integration - all they require is skilled algebra.

1. Absolute and Conditional Convergence

Before we develop the tests, we need a couple of definitions.

Definition 1.1. A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ converges.

There are many series which converge but do not converge absolutely like the alternating harmonic series $\sum (-1)^n/n$ (this converges by the alternating series test). We have a special name for such series.

Definition 1.2. A series $\sum a_n$ is called conditionally convergent if the series converges but it does not converge absolutely.

The following is fairly statement is obvious.

Result 1.3. If a series $\sum a_n$ is absolutely convergent, then it is conditionally convergent.

This gives us a new way to approach series which have positive and negative terms - if we can show that they are absolutely convergent, then they must be convergent. We illustrate with an example.

Example 1.4. Determine whether the

$$\sum \frac{\sin(n) + \cos(n)}{n^2}$$

converges.
Observe that

$$\left| \frac{\sin(n) + \cos(n)}{n^2} \right| \leqslant \frac{2}{n^2} \leqslant \frac{1}{n^2}$$

and we know that $\sum 1/n^2$ converges, so it follows that

$$\sum \frac{\sin(n) + \cos(n)}{n^2}$$

converges absolutely and hence

$$\sum \frac{\sin(n) + \cos(n)}{n^2}$$

must converge.

2. The Ratio and Root Tests

There are two very important tests for absolute convergence. We shall state them and then look at their uses. The first test is useful for many different cases but is particularly useful if there is an $n!$ somewhere in the expression for a_n.

Result 2.1. (The ratio test)

(i) If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$
then the series $\sum a_n$ converges.

(ii) If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$$
then the series $\sum a_n$ diverges.

(iii) If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$$
then test is inconclusive.

The basic idea behind this is to compare the series $\sum a_n$ to a geometric series - such a series converges if and only if the common ratio is less than 1. If the common ratios of the a_n's is less than 1 for large enough n, then ultimately it will act like a convergent geometric sequence. Alternatively, if the common ratios of the a_n's is greater than 1 for large enough n, then ultimately it will act like a divergent geometric sequence.

The following test is convenient when nth powers occur.

Result 2.2. (The root test)

(i) If
$$\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = L < 1$$
then the series $\sum a_n$ converges.

(ii) If
$$\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = L > 1$$
then the series $\sum a_n$ diverges.

(iii) If
$$\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = L = 1$$
then test is inconclusive.

We illustrate how to use these tests through a few examples.

Example 2.3. Determine whether the following series converge or diverge.

(i)

$$\sum n^{1/n}$$

At first glance, it seems like we should apply the root test, but this involves a root, so the root test will just make it more complicated. Observe that in this case, we can apply L'Hopitals rule to the function $f(x) = x^{1/x}$ because it is of type ∞^0. Specifically, we apply L'Hopital to $\ln(x^{1/x}) = \ln(x)/x$. This give, $(1/x)/x \to 0$ as $x \to \infty$, so $x^{1/x} \to e^0 = 1$ as $x \to \infty$. In particular, $n^{1/n} \to 1$ as $n \to \infty$, so the nth term test shows that this series does not converge.

(ii)

$$\sum \frac{1}{(2n)!}$$

We apply the ratio test:

$$\lim_{n\to\infty} \frac{1/((2(n+1))!)}{1/((2n)!)} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

This means that the series converges (though to what we do not know!).

(iii)

$$\sum \frac{(n!)^2}{(2n)!}$$

We apply the ratio test:

$$\lim_{n\to\infty} \frac{((n+1)!)^2/((2(n+1))!)}{(n!)^2/((2n)!)} = \lim_{n\to\infty} \frac{(2n)!((n+1)!)^2}{(2n+2)!(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$

This means that the series converges (though to what we do not know!).

(iv)

$$\sum \frac{2^n}{(n^3 + 1)}$$

Since there is a power of n, we try the root test:

$$\lim_{n\to\infty} \left(\frac{2^n}{(n^3+1)}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{(n^3+1)^{1/n}} = 2$$

(using L'Hopital on the denominator like the first example). Therefore, the root test implies that this series diverges.

(v)
$$\sum (2 - e^{-n})^n$$
We apply the root test:
$$\lim_{n \to \infty} \left((2 - e^{-n})^n \right)^{1/n} = \lim_{n \to \infty} (2 - e^{-n}) = 2$$
so the root test implies that the series diverges.

(vi)
$$\sum \frac{n!}{n^n}$$
This problem involves both a factorial and a power, so we shall try the ratio test:
$$\lim_{n \to \infty} \frac{(n + 1)!/(n + 1)^{n+1}}{n!/n^n} = \lim_{n \to \infty} \frac{(n + 1)! n^n}{n! (n + 1)^{n+1}} = \lim_{n \to \infty} \frac{(n + 1) n^n}{(n + 1)^n (n + 1)}$$
$$= \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e > 1$$
so the ratio test tells us that this series diverges.