Section 11.8: Power Series

1. Power Series

In this section, we consider generalizing the concept of a series. Recall that a series is an infinite sum of numbers

$$\sum a_n.$$

We can talk about whether or not it converges and in some cases, what it converges to. More generally however, instead of considered a series as an infinite sum of numbers, we could consider an infinite sum of an expression in a variable. If we could determine convergence properties of such series, this would allow us to make conclusions about infinitely many series with just numbers simultaneously (by plugging numbers in for the variable). We start with a definition.

Definition 1.1. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

where x is a variable and the c_n's are called the coefficients of the series (note that this series starts from 0, so there is a constant term to this polynomial). More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called a power series centered at a.

If we are given a power series, it does not immediately make sense to talk about convergence because x is a variable. However, if we choose a value of x, say a, then the power series becomes a regular series

$$\sum c_n a^n$$

and we can asks questions about its convergence etc. A power series may converge for some values of x and diverge for others, so it can be viewed as a function whose domain is the set of all numbers for which it converges. This leads us to the following two questions:

Question 1.2. For what values does a power series converge?

Question 1.3. How do we find what values a power series converges for?

Since a power series is essentially a series, we can use most of the same tests we developed for regular series - the only difference is that we must treat x like a variable and determine the values it converges for. We consider a couple of easy examples to illustrate.

Example 1.4. (*i*) For what values of x does the series

$$\sum x^n$$

converge?

For any fixed value of x, this sum is just a geometric series, and we know all the values for which this converges. Specifically,

$$\sum x^n$$

converges if and only if $|x| < 1$.

(*ii*) For what values of x does the series

$$\sum \frac{x^n}{n!}$$

converge?

Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{x^{n+1}(n!)}{x^n(n+1)!} = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} = 0$$

independent of the choice of x. Therefore, this series converges for all values of x.

(*iii*) For what values of x does the series

$$\sum n!x^n$$

converge?

Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{(n+1)!(x-2)^{(n+1)}}{n!(x-2)^n} = \lim_{n\to\infty} (n+1)(x-2)$$

which diverges if $x \neq 2$. Thus this series converges only at $x = 2$.

The last three examples suggest that there are three possibilities for the convergence of a power series - either all choices of x converge, only those on a certain interval converge or only one does. This is true in general as the following result tells us.

Result 1.5. For a given power series

$$\sum c_n(x-a)^n$$

there are only three possibilities:

(*i*) The series converges only when $x = a$.

(*ii*) The series converges for all x.

(*iii*) There is a positive number R called the radius of convergence such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. If $|x-a| = R$ (so $x = R - a$ or $x = a - R$), then the series may converge or diverge.

If the radius of convergence is R and the series is centered around a, we say that the interval $(a-R, a+R)$ is the interval of convergence (where we include the endpoints if the series converges at them). This leads us to the following question:

Question 1.6. How do we find the interval of convergence?

In order to find the interval of convergence of a power series

$$\sum c_n(x-a)^n$$

we proceed as follows:

(i) Observe that if we apply the ratio test, we are evaluating the limit

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n x^n}\right| = |(x-a)| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|.$$

Let L be this limit. In order to converge, the ratio test has to be less than 1, so we need to solve $|(x-a)|L < 1$. Therefore, in order to converge we must have $|(x-a)| \leqslant \frac{1}{L}$. It follows that the radius of convergence will be the reciprocal of the absolute values of the limit

$$\lim_{n\to\infty}\frac{c_{n+1}}{c_n}.$$

(ii) This tells us that if

$$L = \lim_{n\to\infty}\frac{c_{n+1}}{c_n}$$

then

$$\sum c_n(x-a)^n$$

converges on the interval $(a-1/L, a+1/L)$. To finish the problem off, we determine whether the series converges at either of the endpoints, $a-1/L$ and $a+1/L$.

We illustrate with some examples:

Example 1.7. Find the radius and interval of convergence of the following power series.

(i)

$$\sum \frac{x^n}{n3^n}$$

Applying the ratio test we get

$$\lim_{n\to\infty}\frac{\frac{|x^{n+1}|}{(n+1)3^{n+1}}}{\frac{|x^n|}{n3^n}} = \lim_{n\to\infty}\frac{n|x|}{3(n+1)} = \frac{|x|}{3} < 1,$$

so $|x| < 3$. So the radius of convergence is 3. This means that

$$\sum \frac{x^n}{n3^n}$$

converges in the interval $(-3,3)$. We now need to check the endpoints:

$x=3$ gives

$$\sum \frac{3^n}{n3^n} = \sum \frac{1}{n}$$

which is the harmonic series, so diverges.

$x=-3$ gives

$$\sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n}$$

which is the alternating harmonic series, so converges. Therefore, the interval of convergence is $[-3,3)$.

(ii)

$$\sum \frac{(-2)^n}{\sqrt{n}}(x+3)^n$$

Applying the ratio test we get

$$\lim_{n\to\infty} \frac{\frac{|2^{n+1}(x+3)^{n+1}|}{\sqrt{n+1}}}{\frac{|2^n(x+3)^n|}{\sqrt{n}}} = \lim_{n\to\infty} 2|x+3|\sqrt{\frac{n}{n+1}} = 2|x+3| < 1,$$

so $|x+3| < 1/2$. So the radius of convergence is $1/2$. This means that

$$\sum \frac{x^n}{n3^n}$$

converges in the interval $(-3.5,-2.5)$. We now need to check the endpoints:

$x=-2.5$ gives

$$\sum \frac{(-2)^n}{\sqrt{n}}(\frac{1}{2})^n = \sum \frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test.

$x=-3.5$ gives

$$\sum \frac{(-2)^n}{\sqrt{n}}(-\frac{1}{2})^n = \sum \frac{1}{\sqrt{n}}$$

which diverges by the p series test. Thus the interval of convergence is $[-2.5, 3.5)$.

(iii)

$$\sum \frac{n(x-4)^n}{n^3+1}$$

Applying the ratio test we get

$$\lim_{n\to\infty} \frac{\frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1}}{\frac{n|x-4|^n}{n^3+1}} = \lim_{n\to\infty} \frac{(n+1)(n^3+1)|x+4|}{((n+1)^3+1)n} = |x+4| < 1.$$

So the radius of convergence is 1. This means that

$$\sum \frac{n(x-4)^n}{n^3+1}$$

converges in the interval $(3, 5)$. We now need to check the endpoints:

$x = 3$ gives

$$\sum \frac{n(-1)^n}{n^3+1}$$

which converges by the alternating series test.

$x = 5$ gives

$$\sum \frac{n}{n^3+1} \leqslant \frac{1}{n^2}$$

so converges using the p-series test and the comparison test. Thus the interval of convergence is $[3, 5]$.