Section 11.9: Representations of Functions as Power Series

1. Functions we know

Recall that if x is a number with $|x| < 1$, then the series

$$\sum x^n$$

converges. In fact, since this is a geometric series with $|x| < 1$, we know that this sums to

$$\frac{1}{1-x}.$$

This means that provided $|x| < 1$, the function

$$f(x) = \frac{1}{1-x}$$

agrees with the power series

$$\sum x^n$$

(starting from 0). By "agrees", we mean for any chosen value of x which is substituted into the equation, the infinite sum agrees with the value of the series. This leads to two different questions:

Question 1.1. Can this power series be used to construct other power series which are also functions?

Question 1.2. What other functions can be realized as power series?

We shall answer the second question mainly in the next section. The first question we shall answer through a number of examples and by utilizing tools we have developed in Calculus.

Example 1.3. For the following functions, find a power series representation and determine the interval of convergence.

(i)

$$f(x) = \frac{1}{1+x}$$

We know that

$$\frac{1}{1-x} = \sum x^n$$

on the interval $[-1, 1)$. Substituting $-x$ into the equation, we get

$$\frac{1}{1+x} = \sum (-x)^n = \sum (-1)^n x^n.$$

Observe that this converges on the interval $(-1, 1)$.

(*ii*)

$$f(x) = \frac{x}{9+x^2}$$

We know that

$$\frac{1}{1-x} = \sum x^n$$

on the interval $(-1,1)$. Substituting $-x^2/9$ into the equation, we get

$$\frac{1}{1+x^2/9} = \frac{9}{9+x^2} = \sum \left(-\frac{x^2}{9}\right)^n = \sum (-1)^n \left(\frac{x}{3}\right)^{2n}.$$

This is not quite the equation we are after, but if we multiply by $x/9$, we get

$$\frac{x}{9}\frac{9}{9+x^2} = \frac{x}{9+x^2} = \frac{x}{9}\sum (-1)^n \left(\frac{x}{3}\right)^{2n} = \sum (-1)^{n+1} \left(\frac{x}{3}\right)^{2n+1}.$$

Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{\left(\frac{x}{3}\right)^{2n+1}}{\left(\frac{x}{3}\right)^{2n+3}} = \lim_{n\to\infty} \frac{x^2}{9} < 1$$

provided $|x| < 3$, so it converges on the interval $(-3,3)$. At the endpoint $x = 3$, we have

$$\sum (-1)^{n+1} \left(\frac{3}{3}\right)^{2n+1}$$

which diverges and at $x = -9$, we have

$$\sum (-1)^{n+1} \left(\frac{-3}{3}\right)^{2n+1}$$

which diverges. Hence the interval of convergence is $(-3,3)$.

(*iii*)

$$f(x) = \frac{7x-1}{3x^2+2x-1}$$

Using partial fractions, we get

$$f(x) = 2\frac{1}{1+x} - \frac{1}{1-3x}.$$

For the first, we make the substitution $-x$ and for the second, we make the substitution, $3x$, giving

$$2\frac{1}{1+x} - \frac{1}{1-3x} = \sum 2(-x)^n + \sum (3x)^n = \sum (2(-1)^n + 3^n)x^n.$$

The first series converges for x in the interval $(-1,1)$ and the second for $(-1/3,1/3)$, so the sum will converge for x in the interval $(-1/3,1/3)$.

Observe that each of these functions represented as power series were found using simple composition of functions and other elementary algebraic operations. We can ask if there are other elementary operations which can be performed on power series to obtain new ones and in particular, obtain new functions represented as power series. The following answers this question.

Result 1.4. If the power series

$$\sum c_n(x-a)^n$$

has radius of convergence $R \geqslant 0$, then the function defined by

$$f(x) = \sum c_n(x-a)^n$$

is differentiable and integrable on the interval $(a-R, a+R)$ and

(i)

$$f'(x) = \sum nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

(ii)

$$\int f(x)dx = C + \sum \frac{c_n(x-a)^{n+1}}{n} = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \dots$$

where C is some constant.

The radii of convergence are the same for both the integral and derivative, but the behavior at the endpoints may be different.

We illustrate the uses of these operations on power series with some examples.

Example 1.5. Find power series representations for the following functions and the corresponding radii of convergence.

(i)

$$\ln\left(\frac{1+x}{1-x}\right)$$

Observe that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

Also,

$$\frac{d}{dx}\ln(1+x) = \frac{1}{x+1} = \sum(-x)^n$$

and

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x} = -\sum x^n.$$

Thus

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int\sum(-x)^n dx + \int\sum x^n dx$$

$$= \int \sum ((-1)^n x^n + x^n) dx = \int \sum 2x^{2n} dx = C + \sum \frac{2x^{2n+1}}{2n+1}.$$

Observe that $f(0) = 0$, so $C = 0$ giving

$$\ln\left(\frac{1+x}{1-x}\right) = \sum \frac{2x^{2n+1}}{2n+1}.$$

Since the radii of convergence of both of the original series is 1, it follows that the radius of convergence of this series is 1.

(ii)

$$\frac{1}{(1+x)^2}$$

Observe that

$$\frac{d}{dx}\frac{1}{1+x} = \frac{1}{(1+x)^2}$$

and

$$\frac{1}{1+x} = \sum (-x)^n = \sum (-1)^n x^n.$$

Therefore

$$\frac{1}{(1+x)^2} = \frac{d}{dx}\frac{1}{1+x} = \frac{d}{dx}\sum (-1)^n x^n$$

$$= \sum \frac{d}{dx}(-1)^n x^n = \sum \frac{(-1)^n x^{n-1}}{n}.$$

Since the radius of convergence of the original series is 1, it follows that the radius of convergence of this series will also be 1.

Example 1.6. Use power series to approximate the following indefinite integral:

$$\int_0^{1/3} x^2 \arctan(2x) dx$$

First observe that

$$\frac{d}{dx}\arctan(2x) = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2}$$

$$= 2\sum (-4x^2)^n = 2\sum (-1)^n 16^n x^{2n}$$

so

$$\arctan(2x) = 2\sum (-1)^n 16^n \frac{x^{2n+1}}{2n+1}$$

(since when $x = 0$ we have $\arctan(2x) = 0$). Therefore

$$\int_0^{1/3} x^2 \arctan(2x) dx = \int_0^{1/3} x^2 (2\sum (-1)^n 16^n \frac{x^{2n+1}}{2n+1}) dx$$

$$= \int_0^{1/3} (2\sum (-1)^n 16^n \frac{x^{2n+3}}{2n+1}) dx = \left(2\sum (-1)^n 16^n \frac{x^{2n+4}}{(2n+1)(2n+4)}\right)\Bigg|_0^{1/3}$$

$$\cong 2\left(\frac{(1/3)^4}{(1)(4)} - 16\frac{(1/3)^6}{(3)(6)} + 16^2\frac{(1/3)^8}{(5)(8)}\right) = 0.005685.$$