Section 6.1: Arc Length

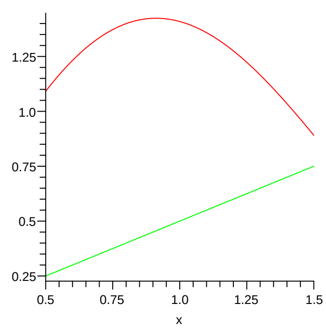
1. Using Riemann Sums to find Area

In this section, we use Riemann sums to determine the area between two different curves. Though you may have already considered this relatively simple idea, we shall still proceed because it is the method which is important and not just the result. We start with the following assumptions:

(i) $[a, b]$ is some finite interval (so a and b are finite real numbers).
(ii) $f(x)$ and $g(x)$ are continuous functions on the interval $[a, b]$.

Question 1.1. How do we find the area bounded between $f(x)$ and $g(x)$ on the interval $[a, b]$?

Answer. We shall use Riemann sums. For ease we shall assume that $f(x) \geqslant g(x)$ for all x in $[a, b]$ (note that this is just for convenience - we shall consider the more general case later). If the graph looks like the following, we proceed as follows:



(i) First note that since one of the graphs is curved, we cannot use elementary geometry to calculate the area. However, we can use elementary geometry to *estimate* the area between the curves using rectangles.
(ii) Break up the interval $[a, b]$ into n equal sized pieces of size $\Delta x = (b - 1)/n$. Choose some point x_1 from the first interval, x_2 from the second and so on, with x_n from the last interval.
(iii) We can approximate the area between f and g in each of these subintervals by using the rectangle whose height is determined by the values of f and g we have chosen in each interval. Specifically, in the 1st, we approximate the area with a rectangle whose top edge has height $f(x_1)$ and whose bottom edge has height $g(x_1)$ and whose length is Δx and so on.

(*iv*) In general, the area of the ith rectangle will be $(f(x_i)-g(x_i))\Delta x$, so summing over all rectangles, we get

$$\text{Area} \sim \sum_{i=1}^{n}(f(x_i)) - g(x_i))\Delta x.$$

(*v*) In order to improve our approximation, we can take smaller and smaller values of Δx noting that as $\Delta x \to 0$ or equivalently as $n \to \infty$, the answer becomes more exact. In particular, we get

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)) - g(x_i))\Delta x.$$

But this looks very familiar!!! Thus we get the following:

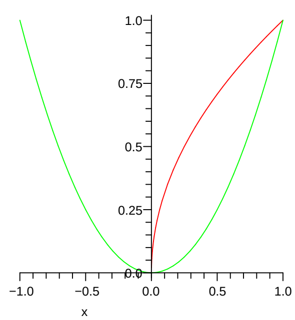
Result 1.2. If $f(x) \geqslant g(x)$ are continuous functions on $[a,b]$, then the area A bounded between f and g on the intevral $[a,b]$ is

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i)) - g(x_i))\Delta x = \int_a^b (f(x) - g(x))dx.$$

To illustrate, we consider a couple of examples.

Example 1.3. Find the area of the **finite** region between $f(x) = \sqrt{x}$ and $g(x) = x^2$.

To answer this question, we first sketch the region:

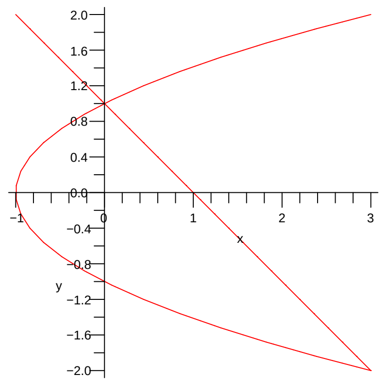


Note that $f(x) \geqslant g(x)$ in the finite region bounded between f and g. Next, to answer the problem, we need to determine the limits of integration. However, the limits are simply where the curves meet, so $x = 0, 1$. Thus we have

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)_0^1 = \frac{1}{3}.$$

Example 1.4. Find the area bounded between $x = y^2-1$ and $x+y = 1$.

To answer this question, we first sketch the region:



Using simple algebra, these two curves meet at the coordinates $(0,1)$ and $(3,-2)$. However, these are not the limits of integration because the upper and lower functions bounding our region differ depending upon the x value. Specifically, for $0 \leqslant x \leqslant 3$, the lower function is $g(x) = -\sqrt{x+1}$ and the upper function is $f(x) = 1-x$. For $-1 \leqslant x \leqslant 0$, the lower function is $g(x) = -\sqrt{x+1}$ and the upper function is $f(x) = \sqrt{x+1}$. Thus to calculate the bounded area, we must evaluate two different integrals and add the two corresponding areas. Specifically, we have:

$$\begin{aligned} A &= \int_{-1}^{0} 2\sqrt{(x+1)}dx + \int_{0}^{3} ((1-x)+\sqrt{(x+1)})dx \\ &= \frac{4}{3}(x+1)^{3/2}\bigg|_{-1}^{0} + \left(x - \frac{x^2}{2} + \frac{2}{3}(x+1)^{3/2}\right)\bigg|_{0}^{3} = \frac{9}{2}. \end{aligned}$$

2. Areas Between General Curves

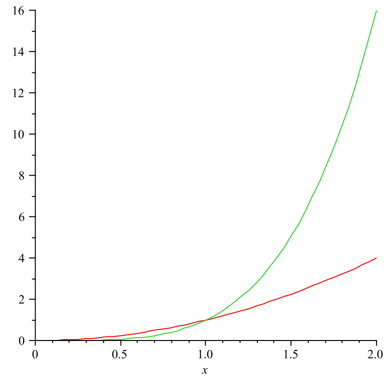
If two curves cross in an interval $[a,b]$, the area cannot be calculated in the same way since the larger of the two functions may change (and the resulting area between $f(x)$ and $g(x)$ would be counted negatively in the integral). The general solution to this is to break up the interval into different smaller intervals on which the functions do not intersect and and then calculate the areas on each of these subintervals using the standard method. The total area will then be the sum of these subareas. Equivalently, we could also just calculate the integral of the absolute value of the difference. We summarize:

Result 2.1. If $f(x)$ and $g(x)$ are continuous on $[a,b]$, then the area bounded between $f(x)$ and $g(x)$ on $[a,b]$ is

$$A = \int_{a}^{b} |f(x) - g(x)|dx.$$

We illustrate with an example.

Example 2.2. Calculate the area bounded x^2 and x^4 on the interval $[0,2]$.



Using simple algebra, these two curves meet at the coordinates $(1, 1)$ and $(3, -2)$. On the interval $[0, 1]$, we have $x^2 \geqslant x^4$ and on the interval $[1, 2]$, we have $x^4 \geqslant x^2$. Thus we need to break the integral up into two parts. Calculating, we get

$$A = \int_0^2 |x^2 - x^4| dx = \int_0^1 (x^2 - x^4) dx + \int_1^2 (x^4 - x^2) dx$$

$$= \left(\frac{x^3}{3} - \frac{x^5}{5} \right)_0^1 + \left(\frac{x^5}{5} - \frac{x^3}{3} \right)_1^2 = \frac{56}{15}.$$

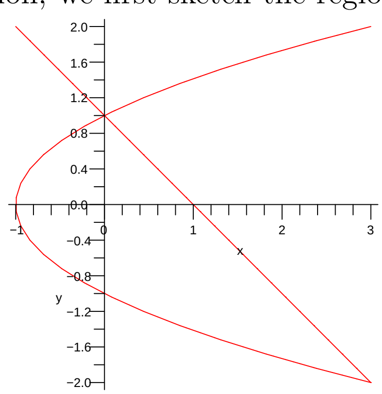
For certain areas, it may be easier to consider x as a function of y rather than y as a function of x. In either case, we get the same method to calculate areas between curves.

Result 2.3. If $f(y)$ and $g(y)$ are continuous for $a \leqslant y \leqslant b$then the area bounded between $f(y)$ and $g(y)$ for $a \leqslant y \leqslant b$ is

$$A = \int_a^b |f(y) - g(y)| dy.$$

Example 2.4. Find the area bounded between $x = y^2 - 1$ and $x + y = 1$.

To answer this question, we first sketch the region:



Instead of using x bounds, we use y bounds. Specifically, the y limits are $-2 \leqslant y \leqslant 1$ with upper y function (the rightmost function) being

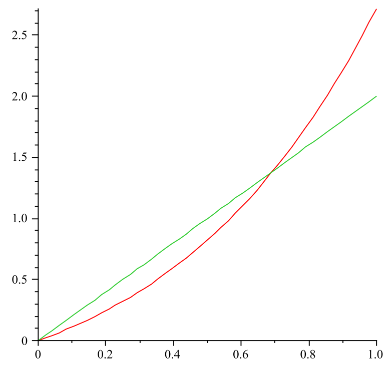
$x = 1 - y$ and lower function (leftmost) being $x = y^2 - 1$. Calculating, we have:

$$A = \int_{-2}^{1} ((1-y) - (y^2 - 1))dy = \int_{-2}^{1} (2 - y - y^2)dy$$
$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1} = \frac{9}{2}.$$

Note that this matches our previous answer but is much easier than the previous case.

Example 2.5. Calculate the area bounded between the functions xe^x and x^2.

To answer this question, we first sketch the region:



Using algebra, we see that the points of intersection are $x = 0$ and $x = \ln(2)$. On the interval $[0, \ln(2)]$, we have $2x \geqslant xe^x$, so to calculate the area bounded between these functions, we need to evaluate the following integral:

$$A = \int_{0}^{\ln(2)} (2x - xe^x)dx.$$

However, we do not know how to evaluate the integral of xe^x, so we need some new integration techniques.

3. Additional Problems

(i) For what values of m do the lime $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of that region.

(ii) Use calculus to find the area of the triangle with vertices $(0, 0)$, $(2, 1)$ and $(-1, 6)$.