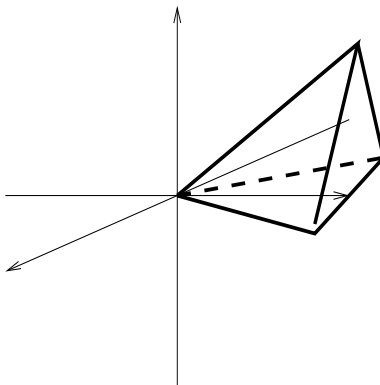


## Section 6.2: Volumes

### 1. VOLUMES USING CROSS SECTION FUNCTIONS

• Have you ever asked yourself how the volume of a sphere or a cone was found? In this section, we use Riemann sums to determine the volumes of certain solids. The idea is to break up the solid into nice easy pieces and then sum up the volume of each of these pieces. The idea is as follows:

- (i) Suppose  $S$  is some solid, sitting in a 3-dimensional plane (the example below is a right triangled bottom with triangle cross sections).
- (ii) As we move along the  $x$ -axis, suppose that the area of the cross section is a function of  $x$ . That is, if we take a plane  $P_x$  perpendicular to the  $x$ -axis through  $x$  and look at the cross section of say  $S$ , then the area of that cross section is a function  $A(x)$ , of  $x$  as illustrated below:



- (iii) We can break up the  $x$ -axis into equal sized intervals of length  $\Delta x$ . If  $x_i$  is some  $x$  value in the  $i$ th interval, then the volume of the solid in that interval can be approximated by the cross-sectional area in that interval multiplied by the length i.e.  $A(x_i)\Delta x$ .
- (iv) The total volume can then be approximated by adding up all these smaller pieces of volume:

$$\text{Volume} \simeq \sum_{i=1}^n A(x_i)\Delta x.$$

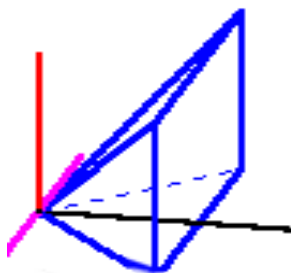
- (v) In order to make the value more exact, we can take smaller and smaller intervals.
- (vi) So we get Area =  $\lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i)\Delta x$ . But this looks very familiar!!! So summarizing, we get:

**Result 1.1.** Let  $S$  be a solid that lies between  $x = a$  and  $x = b$ . If the cross sectional area of  $S$  in the plane  $P_x$  through  $x$  and perpendicular to the  $x$ -axis is  $A(x)$  where  $A$  is a continuous function, then the volume of  $S$  is:

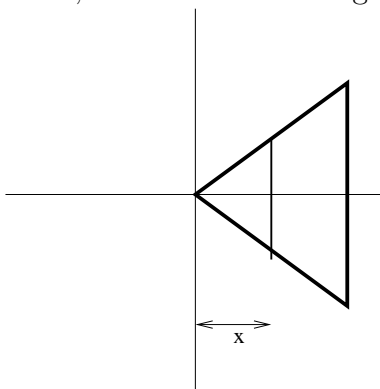
$$V = \int_a^b A(x) dx.$$

**Example 1.2.** Find the volume of the region whose cross sections are squares and whose base is an equilateral triangle centered on the  $x$ -axis where  $0 \leq x \leq 3$ .

We first draw the shape:



Since the cross-sections are squares, it is easy to find their area provided we know the length of one side. However, we know the base is an equilateral triangle, so all sides are equal. This means provided we can find the length of one side, we can find the length of all others.



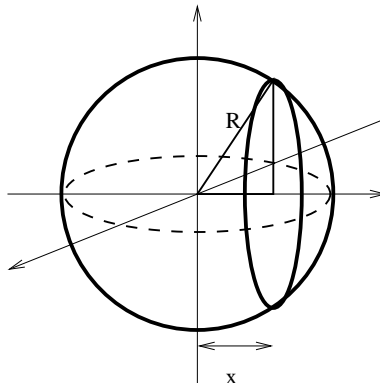
From a simple sketch (see illustration), we see that at the point  $x$ , the length of half of one of the sides is  $x \tan(\pi/6) = \frac{\sqrt{3}x}{3}$ , so the length of one of the sides of the square at  $x$  is  $2\sqrt{3}x/3$ . This means the area of each cross-section will be  $A(x) = \frac{4}{3}x^2$ , so we get

$$V = \int_0^3 \frac{4}{3}x^2 dx = 4x^3 \Big|_0^3 = 108.$$

**Example 1.3.** Show that the volume of a sphere of radius  $R$  is  $\frac{4}{3}\pi R^3$ . Suppose  $S$  is a sphere of radius  $R$ . We place  $S$  in a 3d plane with its center at the origin.

We make some easy observations:

- (i) Then the  $x$  values are between  $-R$  and  $R$ .
- (ii) The cross-sections are circles.
- (iii) At the point  $x$ , the radius of the circle making the cross-section is  $y = \sqrt{(R^2 - x^2)}$  (see illustration).



- (iv) The area of the cross section at  $x$  is  $\pi(R^2 - x^2)$ .

Therefore, we get

$$\begin{aligned}
 V &= \int_{-R}^R \pi(R^2 - x^2)dx = \pi[xR^2 - x^3/3]_{-R}^R \\
 &= \pi[(R^3 - R^3/3) - (-R^3 + R^3/3)] = \pi(2R^3 - 2R^3/3) = 4\pi R^3/3.
 \end{aligned}$$

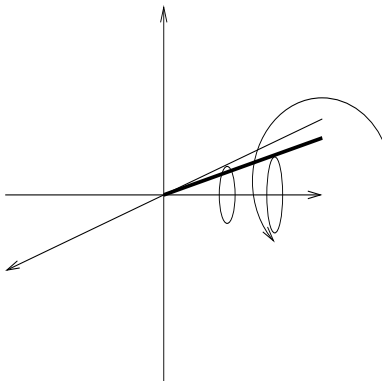
## 2. VOLUMES OF REVOLUTION

We can often form new shapes by rotating two dimensional regions about an axis or a line. If we do this, notice that all cross sections will either be circles or rings, so will be fairly easy to find the area of (provided we know the height of the curves). We illustrate with some well known examples.

**Example 2.1.** Find the volume of a circular cone with height  $H$  and base radius  $R$ .

Observe that this volume can be achieved by rotating a line through the origin outward about the  $x$ -axis (see illustration). The cross sections are circles, so in order to determine their areas, we need to find the radii of each circle. We do this by first noting that the line we need to rotate has the following equation:  $y = \frac{R}{H}(x)$  i.e when  $x = H$ ,  $y = R$  and when  $x = 0$ ,  $y = 0$  (see illustration). Thus, at the point  $x$ , the

radius will be  $\frac{R}{H}x$ . Since the limits are  $0 \leq x \leq H$ , and the area of the cross sections are  $\pi \frac{R^2}{H^2}x^2$ , we get.



$$V = \int_0^H \pi \frac{R^2}{H^2} x^2 dx = \pi \frac{R^2}{H^2} \frac{x^3}{3} \Big|_0^H = \frac{\pi R^2 H}{3}.$$

We do not always have to rotate around the  $x$ -axis and we do not always get disks or circles. In fact, surfaces may be obtained by rotating about any line, and as a result, it may be easier to integrate with respect to  $y$  instead of  $x$  or the slices could be rings instead of circles. We illustrate with examples.

**Example 2.2.** How do we calculate the volume of a ring cake? It can be obtained as a volume of revolution by placing a small thin slice at the appropriate place in the plane and then rotating it about an appropriate axis.

Observe that the cake is bounded 6 different lines. In order to find the volume, we shall have to take integrals over each of the different regions where the cake is bounded. We can calculate the equations of each of the lines using point-slope form. We list the bounds depending on the  $x$  interval.

$[0, 5]$ , the lower function is  $y = -\frac{1}{5}x + 3$  and the upper function is  $y = \frac{1}{6}x + 9$ .

$[5, 6]$ , the lower function is  $y = x - 3$  and the upper function is  $y = \frac{1}{6}x + 9$ .

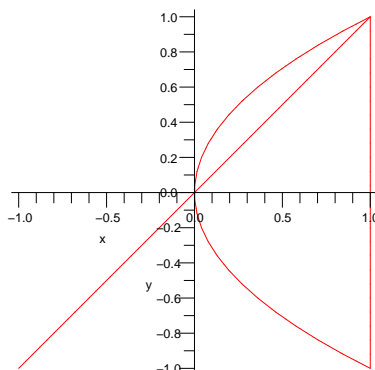
$[6, 7]$ , the lower function is  $y = x - 3$  and the upper function is  $y = -3x + 28$

Since the slices will be washers, we shall have

$$\begin{aligned} V &= \int_0^5 \pi \left( \left( \frac{1}{6}x + 9 \right)^2 - \left( -\frac{1}{5}x + 3 \right)^2 \right) dx \\ &\quad + \int_5^6 \pi \left( \left( \frac{1}{6}x + 9 \right)^2 - (x - 3)^2 \right) dx \\ &\quad + \int_6^7 \pi \left( (-3x + 28)^2 - (x - 3)^2 \right) dx \end{aligned}$$

**Example 2.3.** Find the volume of the solid obtained by rotating the region bounded by  $x = y^2$  and  $x = y$  about the line  $x = 1$ .

First we sketch the region:



These two curves intersect at the points  $(0, 0)$  and  $(1, 1)$ . Since we are rotating about the line  $y = 1$ , the cross-sections with respect to  $y$  will be circular rings. In order to find the area of each cross section, we need to find the radius of each of the rings. However, the outer circle has equation  $x = y^2$  and the inner circle has equation  $x = y$ . This means at height  $y$ , the radius of the outer circle will be  $1 - y^2$  and the inner circle has radius  $1 - y$ . Therefore, the area of a  $y$  cross-section will be  $A(y) = \pi(1 - y^2)^2 - \pi(1 - y)^2 = \pi(y - y^2)$ . Since the  $y$ -values range from 0 to 1, we get

$$V = \int_0^1 \pi(y - y^2) dy = \frac{\pi y^2}{2} - \frac{\pi y^3}{3} \Big|_0^1 = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

We formalize our results.

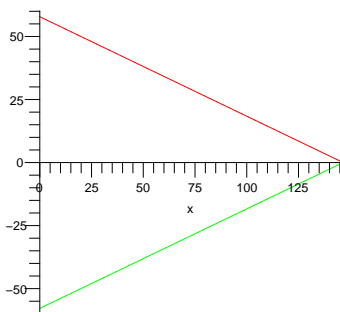
**Result 2.4.** In general, if a surface is obtained by rotating a region about a given line, then the cross sections will either be rings or circles (in terms of either  $x$  or  $y$ ). In order to find the volume of the surface, we still use the integral  $\int_a^b A(x) dx$  or  $\int_a^b A(y) dy$  where  $A(x)$  and  $A(y)$  are the areas of the cross sections. However in this special case,  $A(x)$  and  $A(y)$  are fairly easy to find:

- (i) If the cross sections are circles, then we use  $A = \pi(\text{radius})^2$  where radius is a function of  $x$  or  $y$ .
- (ii) If the cross sections are rings, we find the inner radius  $R_I$  and outer radius  $R_O$  (both functions of either  $x$  or  $y$ ) and we get  $A = \pi(R_O^2 - R_I^2)$ .

Of course, not every surface is obtained as a volume of revolution (in fact, only surfaces which admit a lot of symmetry. We finish by looking at a couple of examples which are not volumes of revolution.

**Example 2.5.** The Pyramids of ancient Egypt are square based. Use Calculus to find the approximate volume of the great Pyramid (height approx 146.59m and base length approx 115.7m).

Observe that if we place the pyramid centered on the  $x$ -axis led horizontally with base at the  $y$ -axis, then the cross sections are squares. So we need to find the lengths of each of these squares with respect to  $x$  in order to find a formula for area of each cross-section. Notice that when  $x = 0$ ,  $y = 115.7/2 = 57.85$  and when  $x = 146.59$ ,  $y = 0$ , see illustration.



Notice that the side of the pyramid follows the line  $57.85 - \frac{57.85}{146.59}x$ , so at the point  $x$ , **half** of the side of the square is  $57.85 - \frac{57.85}{146.59}x$ . Hence  $A(x) = (115.7 - \frac{115.7}{146.59}x)^2 = \frac{(115.7)^2}{(146.59)^2}(146.59 - x)^2$ , so we get

$$\begin{aligned} V &= \int_0^{146.59} \frac{(115.7)^2}{(146.59)^2} (146.59 - x)^2 dx \\ &= -\frac{(115.7)^2}{3 * (146.59)^2} (146.59 - x)^3 \Big|_0^{146.59} = 654109. \end{aligned}$$

**Example 2.6.** Calculate the volume obtained by rotating one arc of the function  $f(x) = \sin(x)$  about the  $x$ -axis.

In this case, the cross-sections are circles, so we can use the results we have developed. Specifically, the bounds on the integral will be  $0 \leq x \leq \pi$  (which is one complete arc of the  $\sin(x)$ ), and the radius is  $f(x) = \sin(x)$ . Thus the volume is equal to the integral

$$V = \int_0^{\pi} \pi \sin^2(x) dx.$$

However, we do not know how to evaluate such an integral, so we need to develop techniques to evaluate trigonometric integrals.