Section 6.3: Volumes by Cylindrical Shells

1. The Method of Cylindrical Shells

Sometimes the area of a given cross section is not easy to find. This means that in some cases, we need to use a different method. One such method is called the method of cyclindrical shells because instead of adding up the volume of thin cross sections with respect to the x-axis, we add up the volume of thin shells in the shapes of hollow cyclinders (like toilet roll holders) which make up the volume. In general, the method of cylindrical shells works as follows:

(i) Suppose S is some solid, obtained by rotating some region around the y-axis.

(ii) As we move along the x-axis, suppose that the height of both the top of the surface and the bottom of the surface is given by functions $h_1(x)$ and $h_2(x)$ respectively and let $h(x) = h_1(x) - h_2(x)$.

(iii) Suppose x_i and x_{i+1} are two points close together with $x_{i+1} > x_i$ and let $\Delta x = x_{i+1} - x_i$.

(iv) The cylindrical shell obtained by rotating the region between x_i and x_{i+1} and bounded by $h_1(x_i)$ and $h_2(x_i)$ is approximately (volume of outer cylinder - volume of inner cylinder)

$$\pi(h(x_i))x_{i+1}^2 - \pi(h(x_i))x_i^2 = \pi(h(x_i))(x_{i+1}^2 - x_i^2)$$

$$= \pi(h)h(x_i)(x_{i+1} + x_i)(x_{i+1} - x_i) = \pi h(x_i)(x_{i+1} + x_i)\Delta x.$$

As the Δx gets smaller, we can approximate x_{i+1} by x_i, so we get $V_S = \pi h(x_i)(x_i + x_i)\Delta x = 2\pi h(x_i)x_i\Delta x$.

(v) Adding up all these cylindrical shells (where the bounds are given by the x-bounds), we get

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi h(x_i)x_i\Delta x.$$

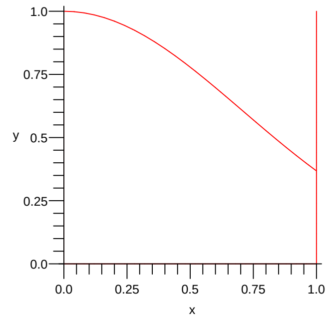
This is just a Riemann sum, so we get the following general result.

Result 1.1. The volume obtained by rotating a region around the y-axis with height function $h(x) = h_1(x) - h_2(x)$ where $a \leqslant x \leqslant b$ is

$$V = \int_a^b 2\pi x h(x)dx.$$

This method is not unique to just rotation about the x-axis and it is clear how to make minor adjustments if we rotate around any other line. We illustrate with some examples.

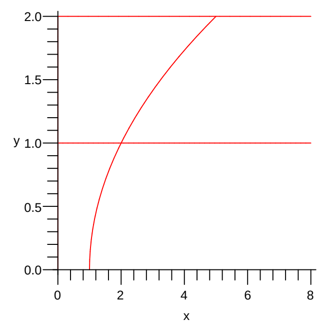
Example 1.2. Find the volume of the solid obtained by rotating $y = e^{-x^2}$ bounded by $y = 0$, $x = 0$ and $x = 1$ about the y-axis.
First we sketch the region:



Here the height of a cylindrical shell is given by the formula $h = e^{-x^2}$ and the bounds are from $x = 0$ to $x = 1$. Therefore we have:

$$V = \int_0^1 2\pi x e^{-x^2} dx = -\pi e^{-x^2}|_0^1 = -\frac{\pi}{e} + \pi = \pi(1 - \frac{1}{e}).$$

Example 1.3. Find the volume of the solid obtained by rotating $x = 1 - y^2$ bounded by $y = 1$ and $y = 2$ about the line x-axis.
First we sketch the region:

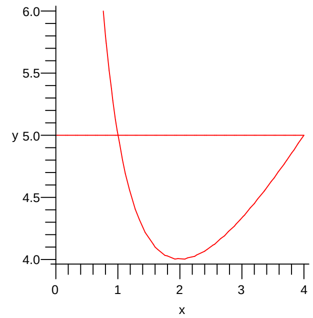


Here we are considering a height function in terms of y. Observe that the bottom has height $y = 0$. The top of the shell has height $x = 1 + y^2$ where $1 \leqslant y \leqslant 2$. Thus using the method of cylindrical shells, we get:

$$V = \int_1^2 2\pi(1 + y^2)y dy = \int_1^2 2\pi(y + y^2)dy = 2\pi(\frac{y^2}{2} + \frac{y^3}{3})|_1^2$$

$$= 2\pi((\frac{4}{2} + \frac{8}{3}) - (\frac{1}{2} + \frac{1}{3})) = \frac{23\pi}{2}.$$

Example 1.4. Find the volume of the solid obtained by rotating $y = x + 4/x$ bounded by $y = 5$ about $x = -1$.
First we sketch the region:



Here we are considering a height function as a function of x, though we have both a top and bottom function. Observe that for a fixed value of x, the upper height will be $h = 5$ and the lower height will be $h = x + 4/x$ so the total height will be $h = 5 - x - 4/x$.
In this case, we cannot use the formula directly. Instead we observe that for a fixed x, the radius r is given by $r = x - (-1) = x + 1$. Since the limits are $1 \leqslant x \leqslant 4$, we get:

$$V = \int_1^4 2\pi(5 - x - 4/x)(x + 1)dx = \int_1^2 2\pi(-x^2 + 4x - \frac{4}{x} + 1)dx$$
$$= 2\pi(-\frac{x^3}{3} + 3x^2 + 4\ln(x) - 9x)|_1^4 = 2\pi(12 - 8\ln(2)).$$

Example 1.5. Find the volume of the surface obtained by rotating the region bounded by the function $f(x) = x\sqrt{1 - x^2}$ and the x-axis with $0 \leqslant x \leqslant 1$ about the y-axis.

This is a straight application of the method of cylindrical shells. The height of the cylindrical shell if $f(x)$ and the base is $y = 0$, so we have

$$V = \int_0^1 2\pi x^2\sqrt{(1 - x^2)}dx.$$

However, we do not know how to evaluate such a function. Thus we need new techniques to approach such problems.

Extra Examples:

- (i) $y = 4x^2$, $2x + y = 6$ about the x-axis.
- (ii) $y = x^2$, $y = 0$, $x = 1$, $x = 2$ about $x = 1$.
- (iii) Use cylindrical shells to find the volume of a sphere with radius R. Is it easier than disks?