

Section 6.4: Work

1. WORK PERFORMED BY A CONSTANT FORCE

Riemann sums are useful in many aspects of mathematics and the physical sciences than just geometry. To illustrate one of its major uses in physics, we consider the problem of calculating the work required to perform a certain task. Before we consider the general problem of calculating work, we first need to consider the problem of calculating work done by a constant force. For this, we need some definitions and basic results from physics.

Definition 1.1. If a constant force of magnitude F (measured in Newtons, $kg * m * s^{-1}$, or measured in pounds, lb) is applied in the direction of motion of an object and if the object moves a distance d (measured in metres, m , or feet, f , then the work performed by the force on the object is

$$W = F * d$$

which is measured in Joules, $J = kg * m * s^{-1}$, if force/distance is measured in Newtons/metres, and foot-pounds, $ft-lb$, if force/distance is measured in pounds/feet.

We illustrate with an example.

Example 1.2. Find the work done in pushing a car a distance of 8m while exerting a constant force of 900N.

Since the force is constant, the work done is simply the product $8 * 900 = 7,200N$.

2. WORK PERFORMED BY A VARIABLE FORCE

The real difficulty when calculating work done is when the force is allowed to vary. In this case however, we can use Riemann sums to help us find work done. Specifically, we have the following:

- (i) Suppose that an object moves along the x -axis in the positive direction from $x = a$ to $x = b$ and at each point x in $[a, b]$, a force $f(x)$ acts on the object where f is a continuous function.
- (ii) Divide the interval $[a, b]$ into n equally sized subintervals of lengths Δx . Choose a point x_1 from the first subinterval, x_2 from the second and so on.
- (iii) The force at the point x_i is $f(x_i)$. Since $f(x)$ is continuous, if n is large, then the force on the i th interval will be approximately constant and so the work done to move the object along the i th interval will be approximated by $f(x_i)\Delta x$.

(iv) It follows that the total work done moving the object from a to b will be approximated by the sum over each of the subintervals:

$$W \sim \sum_{i=1}^n f(x_i)\Delta x.$$

Once again we have a very similar formula, so summarizing, we get:

Result 2.1. The work done by a variable force $f(x)$ to move an object from $x = a$ to $x = b$ where $a \leq x \leq b$ is equal to

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx.$$

We illustrate with an explicit example and then we consider another application.

Example 2.2. A particle is moved along the x -axis by a force which measures $10/(1+x)^2$ at a point x feet from the origin. Find the work done in moving a particle from the origin to a distance of 9ft.

This is a straight application of our result. Specifically, we have

$$W = \int_0^9 \frac{10}{(1+x)^2} dx = -\frac{10}{1+x} \Big|_0^9 = -1 - (-10) = 9ft - lb.$$

An example of a variable force is that one needs to exert to stretch a spring past its natural length. The following tells us how to calculate this force.

Result 2.3. (Hookes Law) The force required to maintain a spring stretched x units beyond its natural length is proportional to x :

$$f(x) = mx$$

for some constant m called the spring constant.

Example 2.4. A force of 6N is required to stretch a string from a natural length of 4m to a length of $4\frac{1}{2}$ m. What work is required to stretch it to a length of 6m.

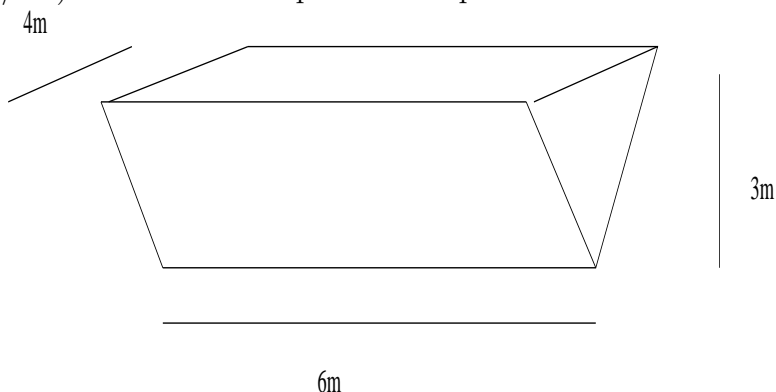
First we need to find the spring constant. Let x denote the distance the spring is stretched beyond its natural length. We know when $x = 1/2$ we have $F = 6$, so it follows that $6 = m/2$ or $m = 12$. Thus the force required to maintain this spring stretched x units beyond its natural length is $f(x) = 12x$. Now we need to calculate the work required to stretch it to a length of 6m. Since we are stretching it from its natural length to 6m, we are finding the work done for $0 \leq x \leq 2$. Therefore we have

$$W = \int_0^2 12x dx = 24J.$$

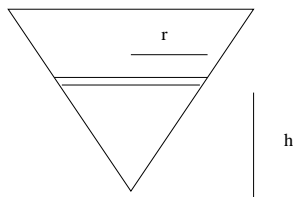
3. OTHER EXAMPLES

The Riemann sums we have considered are for when a variable force moves an object. It could be that a constant force moves different parts of an object different distances (like pulling a rope up the side of a building), or a variable force could be moving different parts of an object different distances (like sucking water out of an odd shaped glass). There are far too many different scenarios to be able to list them all, but in all cases, we can use modified Riemann sums to evaluate the total work done. We finish with a couple of examples.

Example 3.1. Find the work required to pump all the water out of a vat as illustrated below (given that the weight density of water is $9810N/m^3$) if the water is up to 2m deep.



First we note that force is equal to weight density times volume. To calculate the work done in pumping the water out, we first note that the water travels different distances depending upon the depth of the water in the tank. Therefore, we shall slice the water into small horizontal strips of height Δh (we use Δh since h will be the variable measuring the height of the water, so Δh is a measurement of a small change in h , or a slice of water), all of which travel approximately the same distance when being pumped out of the vat and then sum to approximate the total work done. We first make a sketch of the cross section:



First note using similar triangles, a slice of water of height h from the bottom will have radius $2h/3$, so width $4h/3$. This means the volume of a slice of water of height Δh at a height h in the vat will be

$$V = 6 * 4h/3\Delta h = 8h\Delta h.$$

Therefore, the force of gravity on this slice will be

$$78480h\Delta h.$$

Since this slice is at height h , the distance it will be required to move will be $3 - h$, so the total work done to move this small slice of water will be

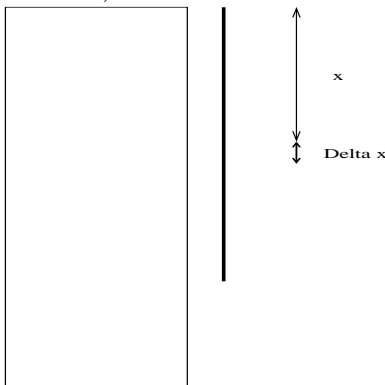
$$78480h(3 - h)\Delta h.$$

Summing over all the strips of water with $0 \leq h \leq 2$ and letting $\Delta h \rightarrow 0$, this sum becomes an integral and we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum 78480h(3 - h)\Delta h &= \int_0^2 78480h(3 - h)dh \\ &= \int_0^2 (235440h - 78480h^2)dh = \left(117720h^2 - 26160h^3\right)_0^2 = 261,600J. \end{aligned}$$

Example 3.2. A 28m uniform chain with a mass of 2kg per meter is dangling from the roof of a building with a bucket with a mass of 25kg on the end. How much work is needed to pull the chain and bucket to the top of the building?

We first consider the work required to pull up just the chain. As in the last example, since different parts of the chain have to move different distances, we shall break the chain up into small pieces, where each small piece travels approximately a fixed distance. Let x denote the distance of a piece of the chain of width Δx from the top of the building, (we use Δx since x is the measurement of distance from the top of the building, so Δx is a measurement of a small change in that distance, or a small piece of the chain). We illustrate.



The force due to gravity on this piece of chain will be $9.8 * 2 * \Delta x = 19.6\Delta x$ (recall that force is equal to mass times acceleration). Since it has to move a distance of x meters, the work required to move this piece of chain will be

$$W = 19.6x\Delta x.$$

Summing up over all the pieces of chain with $0 \leq x \leq 28$ and letting $\Delta x \rightarrow 0$, we get

$$W = \int_0^{28} 19.6x dx = 9.8x^2 \Big|_0^{28} = 7683.2J$$

For the bucket, observe that this is a constant force of $245N$ moving a fixed distance of $28m$, so the total work done is $W = 6860N$. Thus the total work done to move the bucket and chain is $14543.2J$.