Section 6.5: The Average Value of a Function

1. Defining the Average Value of a Function

Suppose that $y_1, \ldots, y_n$ are a set of numbers. In order to find the average value of these numbers, we sum them up and divide by n. That is, Average Value= $\frac{y_1 + \ldots y_n}{n}$. This is useful for discrete mathematics, but usually in mathematics we deal with continuous models (else we cannot apply Calculus). How should we try to find averages for continuous models? Consider the following example.

Example 1.1. Suppose that the temperature $C(t)$ is a function of time in hours, $1 \leqslant t \leqslant 24$. In order to calculate the average temperature over a 24 hour period, we could measure the temperature every hour and then take the average of these measurements:

$$\text{Average Value} \sim \frac{C(1) + C(2) + \cdots + C(24)}{24}.$$

However, this does not take into account that the temperature could suddenly spike or drop at any point. Therefore, to find a better approximation, we should take smaller time intervals. In general, on the interval $[0, 24]$, we could break it up into n intervals, each of length $\Delta x = (b-1)/n = 24/n$. Then we approximate using a time value in each interval and get:

$$\text{Average} \simeq \frac{C(x_1) + \cdots + C(x_n)}{n}.$$

But we know $\Delta x = 24/n$, so we get

$$\text{Average} \simeq \frac{C(x_1) + \cdots + C(x_n)}{n} = \frac{C(x_1) + \cdots + C(x_n)}{24/\Delta x}$$

$$= \frac{(C(x_1) + \cdots + C(x_n))\Delta x}{n} = \frac{1}{24} \sum_{i=1}^{n} C(x_i)\Delta x.$$

This last sum is a Riemann sum, and as we take smaller and smaller subdivisions, we get closer and closer to the definite integral. Thus we could define:

$$\text{Average} \simeq \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} C(x_i)\Delta x = \frac{1}{24} \int_a^b C(x)dx.$$

What we did for this example is not unique to this example. It works in precisely the same way for any continuous function. The only difference we note is that if we are considering the average of a general interval $[a, b]$, we get $\Delta x = (b-a)/n$. In general, we have:

Result 1.2. If $f(x)$ is a continuous function on $[a, b]$, then the average value of f on $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

We illustrate with a couple of examples.

Example 1.3. Find the average values of the following functions over the corresponding intervals.

(i) $1/x$ on $[1, 4]$

$$\text{Average} = \frac{1}{4-1}\int_1^4 \frac{1}{x}dx = \frac{1}{3}\ln(x)|_1^4 = \frac{1}{3}\ln(4).$$

(ii) $3/(1+r)^2$ on $[1, 6]$

$$\text{Average} = \frac{1}{6-1}\int_1^4 \frac{3}{(1+r)^2}dr = -\frac{3}{5+5r}|_1^6 = -\frac{3}{35} + \frac{3}{10} = \frac{3}{14}.$$

Question: Is there a value of f on an interval $[a, b]$ for which $f(c) = $ Average value on $[a, b]$.
Answer: Yes - provided f is a continuous function. This is sometimes known as **the mean value theorem for integrals**:

Result 1.4. If f is continuous on $[a, b]$, then there is a c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

We look at some more applications and examples.

Example 1.5. Suppose that $f(x)$ is a continuous function on $[a, b]$ whose average value on $[a, b]$ is 0. Explain why $f(x)$ must have a zero on $[a, b]$ using the mean value theorem and logical reasoning.

Example 1.6. Suppose $f(x) = (x-3)^2$. Find the average value of $f(x)$ on $[2, 5]$ and the value of c in $[2, 5]$, such that f at this point equals that value.

$$\text{Average} = \frac{1}{5-2}\int_2^5 (x-3)^2 dx = \frac{(x-3)^3}{9}|_2^5 = \frac{8}{9} + \frac{1}{9} = 1.$$

Then $(x-3)^2 = 1$ implies $x - 3 = 1$ or $x - 3 = -1$, so $x = 4$ or $x = 2$ (observe in this case there are two values!).

Example 1.7. (i) What is the average value of $y = $ on $[0, 4]$.

$$\text{Average} = \frac{1}{4}\int_0^4 x^3 dx = \frac{x^4}{4}|_0^4 = 64.$$

(ii) Which is bigger, the rectangle of height 64 and width 4 or the area under x^4 between 0 and 4?