# Section 7.1: Integration by Parts

## 1. Introduction to Integration Techniques

Unlike differentiation where there are a large number of rules which allow you (in principle) to differentiate any function, the operation of integration is much more technical and there are in fact many functions for which there just don't exist algebraic antiderivatives (take $\frac{\sin(x)}{x}$ for example, the TI-89 just gives you back the same function suggesting that there is in fact no algebraic antiderivative for this function). There are two ways we are going to try to further our understandings and abilities in integration.

($i$) Derive rules and techniques, like we did with derivatives, to allow us to integrate as many different types of functions as possible.

($ii$) Look at different ways to write functions down for which it may be easier to find integrals (linear approximations etc).

The majority of Chapter 7 focuses on deriving new rules and techniques for integration (Chapter 11 will focus on the problem of realizing functions in new ways). We start by briefly recalling substitution which arose as an attempt to reverse the chain rule.

## 2. Review of Integration by Substitution

**Result 2.1.** If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Recall that the basic approach to problems in integration by substitution are as follows:

($i$) Identify an "inside function" $g(x)$ and an "outside function" $f(x)$ so that the integrand looks something like $f(g(x))g'(x)$

($ii$) Set $u = g(x)$ and then

$$\frac{du}{dx} = g'(x),$$

or

$$dx = \frac{1}{g'(x)}du.$$

In the original integrand, replace $g(x)$ by $u$ and $dx$ by $\frac{1}{g'(x)}du$.

($iii$) Simplify the equation. Provided no mistakes have been made, you should have an integrand solely in terms of $u$. If not, either a mistake has been made or the substitution you have used does not work or there may be further simplifications which can be used to eliminate $x$ from the equation.

($iv$) Evaluate the integral as an integral with respect to $u$.

($v$) Substitute back in for $x$.

We illustrate the process with a couple of examples:

**Example 2.2.** Evaluate the following integrals.

($i$) $\int \frac{1+4x}{\sqrt{(1+x+2x^2)}}dx$

Choose $u = 1 + x + 2x^2$, then $du/dx = 1 + 4x$, or $dx = du/(1 + 4x)$. Making the substitution, we get

$$\int \frac{1+4x}{\sqrt{(1+x+2x^2)}}dx = \int \frac{1+4x}{\sqrt{u}}\frac{1}{1+4x}du = \int \frac{1}{\sqrt{u}}du$$

$$= \int u^{-1/2}du = 2u^{1/2} + C = 2\sqrt{(1+x+2x^2)} + C.$$

($ii$) $\int x\sqrt{x+1}dx$

Choose $u = 1 + x$, then $du/dx = 1$ or $du = dx$. Making the substitution, we get

$$\int x\sqrt{u}du$$

which looks like something we still cannot integrate. However, $u = x + 1$, so $x = u - 1$ giving,

$$\int x\sqrt{u}du = \int (u-1)u^{1/2}du = \int u^{3/2} - u^{1/2}du$$

$$= \frac{2}{5}u^{5/2} - \frac{3}{2}u^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{3}{2}(x+1)^{3/2} + C.$$

**Question 2.3.** The idea of substitution came from trying to reverse the chain rule. Are there any other rules we can try to reverse?

## 3. Integration by Parts

Recall, the product rule says the following: if $f(x)$ and $g(x)$ are differentiable functions, then the derivative of $f(x)g(x)$ is $f'(x)g(x)+f(x)g'(x)$ i.e

$$\frac{d}{dx}f(x)g(x) = (\frac{d}{dx}f(x))g(x) + f(x)(\frac{d}{dx}g(x)).$$

This means that

$$\frac{d}{dx}f(x)g(x) - (\frac{d}{dx}f(x))g(x) = f(x)(\frac{d}{dx}g(x)).$$

If we integrate both sides and apply the fundamental theorem of calculus, we get,

$$\int \frac{d}{dx}(f(x)g(x))) - (\frac{d}{dx}f(x))g(x))dx = \int (f(x)(\frac{d}{dx}g(x)))dx.$$

But,

$$\int \frac{d}{dx}(f(x)g(x)) - (\frac{d}{dx}f(x))g(x)dx$$
$$= f(x)g(x) - \int (\frac{d}{dx}f(x))g(x)dx = \int (f(x)(\frac{d}{dx}g(x)))dx.$$

This means in order to calculate the integral of a function $f(x)$ multiplied by a derivative of a function $g'(x)$, we can apply this formula i.e

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

This is not quite a product rule, but it can be very handy if we have a product of functions, one of which has an easy integral and the other an easy derivative. We shall look at the process of integration by parts in detail.

($i$) In the integrand, look for a product of functions. Set $u = f(x)$, a function whose derivative is easier than $f(x)$, and set $\frac{dv}{dx} = g'(x)$, a function for which it is easy to find an antiderivative. If in doubt, use LIPET (logarithm, Inverse Trip, Polynomial, Exponential, Trig Function) in your choice for $f(x)$.

($ii$) Find $v = \int g'(x)dx$ and $\frac{du}{dx} = f'(x)$.

($iii$) To calculate $\int f(x)g'(x)dx$ we use the formula $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. in terms of $u$ and $v$, we have

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

The integral on the right hand side in principle should be easier than the integral on the left hand side, so we integrate.

We illustrate the process with a number of examples.

**Example 3.1.** Use integration by parts to evaluate the following integrals.

($i$) $\int xe^x dx$

Here the function $x$ will become easier if differentiated, so we set $u = x$ and $dv/dx = e^x$. Then, $du/dx = 1$ and $v = e^x$. Applying the formula, we get,

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

($ii$) $\int x\cos(5x)dx$

Here the function $x$ will become much easier if differentiated, so we set $u = x$ and $dv/dx = \cos(5x)$. Then $du/dx = 1$ and $v = (1/5)\sin(5x)$. Applying integration by parts, we get

$$\int x\cos(5x)dx = \frac{x}{5}\sin(5x) - \int \frac{1}{5}\sin(5x)dx$$

$$= \frac{x}{5}\sin(5x) + \frac{1}{25}\cos(5x) + C.$$

(*iii*) $\int \ln(x)dx$

Here it does not look like the product of two functions. However, notice that any function $f(x)$ is equal to $1 * f(x)$. In this case, we can eliminate $\ln(x)$ through differentiation, so we set $u = \ln(x)$ and $dv/dx = 1$ giving, $du/dx = 1/x$ and $v = x$. Then applying the formula, we get

$$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - x + C.$$

(*iv*) $\int \sin^{-1}(x)dx$.

This question is similar to the previous one. We set $u = \sin^{-1}(x)$ and $dv/dx = 1$ giving, $du/dx = 1/\sqrt{1-x^2}$ and $v = x$. Then applying the formula, we get

$$\int \sin^{-1}(x)dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx.$$

To solve this last integral, we need to use substitution. We take $u = 1 - x^2$, then $du/dx = -2x$ or $dx = du/(-2x)$ giving

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{u}}\frac{-1}{2x}du = -\frac{1}{2}\int u^{-1/2}du = -u^{1/2} = -\sqrt{1-x^2}.$$

Putting these together, we get

$$\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C.$$

(*v*) $\int x^4(\ln(x))^2dx$

Here we cannot integrate $(\ln(x))^2$, so we shall set $u = (\ln(x))^2$ and $dv/dx = x^4$. Then $du/dx = 2\ln(x)/x$ and $v = x^5/5$. Applying the formula, we get:

$$\int x^4(\ln(x))^2dx = \left[(\ln(x))^2\frac{x^5}{5} - \int \frac{2\ln(x)x^5}{5x}dx\right]$$

$$= \left[(\ln(x))^2\frac{x^5}{5} - \frac{2}{5}\int \ln(x)x^4dx\right]$$

This second integral is also not possible without integration by parts, so again we set $dv/dx = x^4$ and $u = \ln(x)$, so $v = x^5/5$ and $du/dx = 1/x$. Applying the formula, we get:

$$\int x^4\ln(x)dx = \left[\ln(x)\frac{x^5}{5} - \int \frac{x^5}{5x}dx\right]$$

$$= \left[\ln(x)\frac{x^5}{5} - \frac{1}{5}\int x^4dx\right] = \left[\ln(x)\frac{x^5}{5} - \frac{x^5}{25}\right]$$

Putting these two together, we get

$$\int x^4(\ln(x))^2dx = \left[(\ln(x))^2\frac{x^5}{5} - \frac{2}{5}\int \ln(x)x^4dx\right]$$

$$\left[(\ln(x))^2\frac{x^5}{5} - \frac{2}{5}\left[\ln(x)\frac{x^5}{5} - \frac{x^5}{25}\right]\right]$$

$$= \left[(\ln(x))^2\frac{x^5}{5} - \ln(x)\frac{2x^5}{25} + \frac{2x^5}{125}\right]$$

## 4. Reduction Formulas

The last example we considered required performing integration by parts twice. In general, when a function involves large powers, in practice, integration by parts can be used to derive an antiderivative. However, in many cases it is not practical. What is practical however is finding instead a formula which one can use a number of times rather than following the same process continually. Finding a formula using integration by parts which reduces the complexity of an integral without actually solving it is called finding a **reduction formula**. We illustrate by example.

**Example 4.1.** Show that $\int(\ln(x))^n dx = x(\ln(x))^n - n\int(\ln(x))^{n-1}dx$. Why does this help?
Even though we cannot fully evaluate it, we shall use integration by parts. We choose $u = (\ln(x))^n$ and $dv/dx = 1$ giving $du/dx = n(\ln(x))^{n-1}/x$ and $v = x$. Applying the integration by parts formula, we get

$$\int(\ln(x))^n dx = x(\ln(x))^n - \int \frac{n(\ln(x))^{n-1}}{x}xdx$$

$$= x(\ln(x))^n - n\int(\ln(x))^{n-1}dx.$$

This formula is helpful because it allows us to evaluate such integrals without having to apply integration by parts continually. For example, if $n = 3$, then

$$\int(\ln(x))^3dx = x(\ln(x))^3 - 3\int(\ln(x))^2dx$$

$$= x(\ln(x))^3 - 3\left[x(\ln(x))^2 - 2\int(\ln(x))^1dx\right]$$

$$= x(\ln(x))^3 - 3\left[x(\ln(x))^2 - 2(x\ln(x) - x)\right].$$

## 5. Definite Integrals

As with substitution, integration by parts can be used to evaluate definite integrals in exactly he same way using the fundamental theorem of calculus.

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x) - \int f(x)g'(x)dx\right]_a^b$$

We illustrate with an example.

**Example 5.1.** Evaluate the definite integral $\int_0^1 x5^x dx$.
In this case, we take $u = x$ and $dv/dx = 5^x$. Then we have, $du/dx = 1$ and $v = 5^x/\ln(5)$. Applying integration by parts, we have

$$\int_0^1 x5^x dx = \left[\frac{x5^x}{\ln(x)} - \int \frac{5^x}{\ln(5)} dx\right]_0^1$$

$$= \left[\frac{x5^x}{\ln(5)} - \frac{5^x}{(\ln(5))^2}\right]_0^1 = \frac{5}{\ln(5)} - \frac{4}{\ln(5)^2}$$