Section 7.3: Trigonometric Substitution

1. Types of Trigonometric Substitutions

We have seen how to evaluate the definite integral $\int_{-1}^{1} \sqrt{1-x^2}dx$ by interpreting it as the area of a half circle. We would like, however, to be able to find an antiderivative for this function, and other functions like this, which involve square roots and squares of numbers. The basic idea behind such integrals is to choose a convenient substitution which cancels with enough things to give us something integrable. We start by looking at an example.

Example 1.1. Find the integral $\int \sqrt{1-x^2}dx$.
To solve this problem, we use the substitution $x = \sin(u)$. Then $dx/du = \cos(u)$ or $dx = \cos(u)du$. Then

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2(u)}\cos(u)du = \int \cos^2(u)du$$

provided $-\pi/2 \leqslant u \leqslant \pi/2$. Then using the half angle identity, we get

$$\int \cos^2(u)dx = \int \frac{1}{2}(1+\cos(2u))dx = \frac{u}{2} + \frac{\sin(2u)}{4} + C.$$

Observe however that the original function was in terms of x, so we need to simplify and substitute back in to get everything in terms of x. Using the double angle formula, we get

$$\frac{u}{2} + \frac{\sin(2u)}{4} + C = \frac{u}{2} + \frac{\sin(u)\cos(u)}{2} + C.$$

Since $x = \sin(u)$, it follows that $\cos(u) = \sqrt{1-x^2}$, so

$$\frac{\arcsin(x)}{2} + \frac{x\sqrt{1-x^2}}{2} + C = .$$

By making this simple trigonometric substitution, we were able to eliminate the square root through the use of trigonometric identities and then use trigonometric identities and pythagoras to simplify. For a general expression involving square roots and squares, the substitution $x = \cos(u)$ may not work. However, other trigonometric substitutions may work. We summarize the three main equation types for which trigonometric substitution will work.

Result 1.2. The following is a list of which trigonometric substitutions to make under which circumstances.

- (i) For expressions involving $\sqrt{a^2-x^2}$ where a is a constant, substitute $x = a\cos(u)$, where $-\pi/2 \leqslant u \leqslant \pi/2$, simplify and then use trigonometric integration.
- (ii) For expressions involving $\sqrt{x^2+a^2}$ where a is a constant, substitute $x = a\tan(u)$, where $-\pi/2 \leqslant u \leqslant \pi/2$, simplify and then use trigonometric integration.

(*iii*) For expressions involving $\sqrt{x^2 - a^2}$ where a is a constant, substitute $x = a\sec(u)$, where $0 \leqslant u \leqslant \pi/2$ or $\pi \leqslant u \leqslant 3\pi/2$, simplify and then use trigonometric integration.

We illustrate with an example of each. Since we have already considered the first, we just focus on the second two.

Example 1.3. Evaluate
$$\int \frac{1}{\sqrt{x^2+1}}dx.$$
We make the substitution $x = \tan(u)$, so $dx = \sec^2(u)du$. Then,
$$\int \frac{1}{\sqrt{x^2+1}}dx = \int \frac{1}{\sqrt{(\tan^2(u)+1)}}\sec^2(u)du$$
$$= \int \sec(u)du = \ln|\sec(u) + \tan(u)| + C.$$
Since the original equation was in terms of x, we need to get the answer in terms of x. However, since $x = \tan(u)$, we must have $\sec(u) = \sqrt{1+x^2}$, so
$$\int \frac{1}{\sqrt{x^2+1}}dx = \ln|\sqrt{1+x^2} + x| + C.$$
Example 1.4. Evaluate
$$\int \frac{\sqrt{x^2-9}}{x^3}dx.$$
Let $x = 3\sec(u)$ so $dx = 3\sec(u)\tan(u)du$. Then we have
$$\int \frac{\sqrt{x^2-9}}{x^3}dx = \int \frac{\sqrt{9\sec^2(u)-9}}{(3\sec(u))^3}3\tan(u)\sec(u)du$$
$$= \frac{1}{3}\int \frac{\tan(u)}{\sec^3(u)}\sec(u)\tan(u)du = \frac{1}{3}\int \frac{\tan^2(u)}{\sec^2(u)}du = \frac{1}{3}\int \sin^2(u)du.$$
This last integral can be evaluated using the half angle identities, and we can simplify using the double angle formula. We get
$$\frac{1}{3}\int \sin^2(u)du = \frac{1}{3}\left(\frac{u}{2} - \frac{\sin(2u)}{4}\right) + C = \frac{1}{3}\left(\frac{u}{2} - \frac{\sin(u)\cos(u)}{2}\right) + C.$$
Finally, if $x = 3\sec(u)$, then $\sec(u) = x/3$. Consequently $\cos(u) = 3/x$ and $\sin(u) = \sqrt{1 - 9/x^2}$ so we get
$$\frac{1}{3}\left(\frac{u}{2} - \frac{\sin(u)\cos(u)}{2}\right) + C = \frac{\sec^{-1}(x)}{6} + \frac{\sqrt{1-9/x^2}}{2x} + C$$
$$= \frac{\sec^{-1}(x)}{6} + \frac{\sqrt{x^2-9}}{2x^2} + C.$$