Section 7.4: Integration of Rational Functions by Partial Fractions

"This is about as complicated as it gets"

1. The Method of Partial Fractions

Except for a few very special cases, currently we have no way to find the integral of a general rational function. In this section we shall solve this problem. In practicality, the method we shall develop is long and cumbersome, but the most important thing is that, in general, it will always work (though we may not always want to do it!). Before we start with the integration, we need to develop a method of reducing a rational function called the method of partial fractions. We motivate our actions with an example.

Example 1.1. Evaluate the following integrals.

(i)

$$\int x + \frac{1}{x-1} + \frac{1}{x+1} dx$$

For this integral we just break it up and evaluate each piece individually:

$$\begin{aligned}\int x + \frac{1}{x-1} + \frac{1}{x+1} dx &= \int x dx + \int \frac{1}{x-1} dx + \int \frac{1}{x+1} dx \\ &= \frac{x^2}{2} + \ln(x-1) + \ln(x+1) + C\end{aligned}$$

(ii)

$$\int \frac{x(x^2+1)}{(x^2-1)} dx$$

Currently we have no way to evaluate this integral. However, notice that

$$\frac{x(x^2+1)}{(x^2-1)} = x + \frac{1}{x-1} + \frac{1}{x+1}$$

if you find a common denominator, and we can integrate this. Thus

$$\frac{x(x^2+1)}{(x^2-1)} dx == \frac{x^2}{2} + \ln(x-1) + \ln(x+1) + C.$$

By the last example, it looks like if we can somehow break a fraction down by reversing the process of finding a common denominator, we should be able to integrate certain rational functions. The method of "reversing finding a common denominator" is a very important tool in Calculus, so it has its own name - The Method of Partial Fractions.

Suppose that $R(x) = s(x)/q(x)$ is a rational functions where $s(x)$ and $q(x)$ are polynomials. The method of partial fractions is fairly difficult, so we start by describing the general set up and then proceed through the different cases. We always start by doing the following:

(i) If the degree of $s(x)$ is greater or equal to the degree of $q(x)$, use polynomial division to reduce $R(x)$ to an expression of the form $R(x) = t(x) + p(x)/q(x)$ where $t(x)$ is a polynomial of the degree of p is less than the degree of q.

(ii) Factor $q(x)$ completely. It is a fact from algebra that $q(x)$ will factor combinations of powers of linear functions (things of the form $(x-a)^n$) and powers of quadratics (things of the form $(ax^2+bx+c)^m$.

We illustrate these first two steps with an example.

Example 1.2.

$$\frac{3x^3-2x+2}{x^3-3x^+2x} = 3 + \frac{9x^2-8x+2}{x^3-3x^2+2x} = 3 + \frac{9x^2-8x+2}{x(x-2)(x-1)}$$

$$\begin{array}{r|l} & \quad 3 \\ \hline x^3-3x^2+2x\,) & 3x^3 \qquad\quad -2x+2 \\ & -3x^3+9x^2-6x \\ \hline & \qquad 9x^2-8x \end{array}$$

The next step depends upon how $q(x)$ factors. We shall state the main theorem below and then for ease, we break up the different possibilities and consider each of them in turn. First, the main result of partial fractions is the following:

Result 1.3. Suppose that $R(x) = p(x)/q(x)$ where the degree of p is less than the degree of q. Then the following are true:

(i) $q(x)$ factors into a product of linear and quadratic factors where the quadratic factors have no real roots.

(ii) Each factor $(x+a)^n$ of $q(x)$ contributes

$$\frac{A_1}{(x+a)} + \frac{A_2}{(x+a)^2} + \dots \frac{A_{n-1}}{(x+a)^{n-1}}$$

to partial fractions.

(iii) Each factor $(x^2+ax+b)^n$ of $q(x)$ contributes

$$\frac{A_1x+B_1}{(x^2+ax+b)} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \dots \frac{A_{n-1}x+B_{n-1}}{(x^2+ax+b)^{n-1}}$$

to partial fractions.

To find the A_i and B_i, we write $R(x)$ equal to its possible partial fraction representation, cross-multiply to clear denominators and then choose appropriate values of x to substitute in.

2. $q(x)$ FACTORS INTO DISTINCT LINEAR TERMS

In this case, it is a fact from algebra that if $q(x) = (x - a_1)(x - a_2)\dots(x - a_n)$ where all of the a_i are different, then

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x-a_1)} + \cdots + \frac{A_n}{(x-a_n)}$$

(so $p(x)/q(x)$ can be written as a sum of reciprocals of linear factors). In order to calculate the numbers $A_1, \dots, A_n$, we cross multiply both sides of the equation by the denominators, and then substitute in the values $a_1, \dots, a_n$ which will allow us to ultimately solve for $A_1, \dots, A_n$. We illustrate by example.

Example 2.1. Recall

$$\frac{3x^3 - 2x + 2}{x^3 - 3x^2 + 2x} = 3 + \frac{9x^2 - 8x + 2}{x^3 - 3x^2 + 2x} = 3 + \frac{9x^2 - 8x + 2}{x(x-2)(x-1)}$$

It follows that

$$\frac{9x^2 - 8x + 2}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}.$$

Clearing denominators, we get

$$9x^2 - 8x + 2 = A(x-2)(x-1) + Bx(x-1) + Cx(x-2).$$

Substituting in $x = 0$, we get $1 = A(-2)(-1) = 2A$, so $A = 1$. Substituting $x = 1$, we get $3 = -C$ or $C = -3$. Finally, substituting $x = 2$, we get $22 = 2B$ or $B = 11$. Thus we get

$$\frac{3x^3 - 2x + 2}{x^3 - 3x^2 + 2x} = 3 + \frac{1}{x} + \frac{11}{x-2} - \frac{3}{x-1}.$$

Using this method of partial fractions, we can now integrate any function of this form as follows:

Example 2.2.

$$\int \frac{3x^3 - 2x + 2}{x^3 - 3x^2 + 2x} dx = \int 3 + \frac{1}{x} + \frac{11}{x-2} - \frac{3}{x-1} dx$$
$$= 3x + \ln(x) + 11\ln(x-2) - 3\ln(x-1) + C$$

3. $q(x)$ IS A PRODUCT OF LINEAR FACTORS

$q(x)$ is a product of linear factors, some of which are repeated (so $q(x) = (x - a_1)^{n_1} \dots (x - a_r)^{n_r}$). This case is similar to the last, the only difference is the fact that we need to take into account that larger powers of linear factors can occur. For ease, we discuss how to evaluate each linear factor.

It is a fact from algebra that if $(x-a)^r)$ is a linear factor of $q(x)$, then when reduced into partial fractions, $p(x)/q(x)$ will contain terms of the form

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_r}{(x-a)^r}.$$

Once this is done, we can use logarithms or substitution and the power rule to evaluate the integrals. Again, to show how this works, we illustrate by example.

Example 3.1. Find

$$\int \frac{x^2}{(x-3)(x-2)^2} dx$$

First we observe that

$$\frac{x^2}{(x-3)(x-2)^2} = \frac{A}{(x-3)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Clearing denominators, we get

$$x^2 = A(x-2)^2 + B(x-2)(x-3) + C(x-3).$$

Substituting $x = 2$ gives $4 = -C$ or $C = -4$, and substituting $x = 3$, we get $9 = A$. To find B, we substitute $x = 0$ giving, $0 = 9*4+6B+4*3$ or $B = -8$. Thus

$$\frac{x^2}{(x-3)(x-2)^2} = \frac{9}{(x-3)} - \frac{8}{(x-2)} - \frac{4}{(x-2)^2}.$$

Therefore,

$$\int \frac{x^2}{(x-3)(x-2)^2} dx = \int \frac{9}{(x-3)} - \frac{8}{(x-2)} - \frac{4}{(x-2)^2} dx$$
$$= 9\ln(x-3) - 8\ln(x-2) + \frac{4}{(x-2)} + C$$

Now (in principle) we can integrate any rational function provided the factors of the denominator $q(x)$ are only linear. The only other possibility is that there are quadratic factors in $q(x)$ which will not reduce. We consider this case now.

4. $q(x)$ contains non-reducible quadratic factors, none of which are repeated

If $ax^2 + bx + c$ is one such factor, then the fraction

$$\frac{Ax+B}{ax^2+bx+c}$$

appears in the partial fraction expansion for $R(x)$ (note that we must have $b^2 - 4ac < 0$, else it has roots and can be factored). The method of partial fractions is identical as with other cases, though when it comes to evaluating the integral, it is a little more complicated. We illustrate with a couple of examples, first how to perform the integrals and secondly how to evaluate the partial fractions and perform the integrals.

Example 4.1. How do we integrate

$$\frac{Ax+B}{ax^2+bx+c}$$

if $b^2-4ac<0$? We complete the square and use a special kind of substitution.
First observe that

$$\frac{Ax+B}{ax^2+bx+c}=\frac{1}{a}\frac{Ax+B}{x^2+bx/a+c/a},$$

so since integrals are linear over constant multiples, we may assume $a=1$, so we are trying to integrate

$$\frac{Ax+B}{x^2+bx+c}.$$

Completeing the square on the denominator, we get

$$\frac{Ax+B}{(x+b/2)^2-b^2/4+c}.$$

We make the substitution $u=x+b/2$, so $x=u-b/2$, giving $du=dx$. Then,

$$\int\frac{Ax+B}{(x+b/2)^2-b^2/4+c}dx=\int\frac{A(u-b/2)+B}{u^2+(c-b^2/4)}du$$

$$=\int\frac{Au+(B-Ab/2)}{u^2+(c-b^2/4)}du=A\int\frac{u}{u^2+(c-b^2/4)}du+\int\frac{B-Ab/2}{u^2+(c-b^2/4)}du.$$

Observe that we can integrate both of these integrals. In particular, one is a logarithm and the other is an inverse tangent. That is:

$$A\int\frac{u}{u^2+(c-b^2/4)}du=\frac{A\ln(u^2+(c-b^2/4))}{2}$$

and

$$=\frac{A\ln((x+b/2)^2+(c-b^2/4))}{2},$$

and

$$\int\frac{B-Ab/2}{u^2+(c-b^2/4)}$$

is an inverse tangent (though more complicated, so instead we shall look at examples!).

Example 4.2. Find the integral

$$\int \frac{x}{x^2+4x+8}dx.$$

We first observe that $b^2 - 4ac = 16 - 32 < 0$, so the denominator is an irreducible quadratic, so the method of partial fractions is not required. In order to integrate a function of this form, we need to use substitution and the method of completing the square. We use the following protocol:

(*i*) Complete the square of the denominator: $x^2 + 4x + 8 = (x + 2)^2 - 4 + 8 = (x + 2)^2 + 4$.

(*ii*) Make the substitution $u = x + 2$, so $du = dx$ and $x = u - 2$:

$$\int \frac{x}{x^2+4x+8}dx = \int \frac{u-2}{u^2+4}du$$

(*iii*) Break the fraction up into two different integrals

$$\int \frac{u}{u^2+4}du - \int \frac{2}{u^2+4}du.$$

The first integral is a logarithmic substitution, the second is an inverse tangent substitution.

(*iv*) For the first integral, we have

$$\int \frac{u}{u^2+4}du = \frac{\ln(u^2+4)}{2}.$$

(*v*) For the second integral, we have

$$-2\int \frac{1}{u^2+4}du,$$

so we set $v = u/2$ so $du = 2dv$, giving

$$-2\int \frac{1}{u^2+4}du = -4\int \frac{1}{4v^2+4}dv = -\int \frac{1}{v^2+1}$$
$$= -\arctan(v) = -\arctan(u/2)$$

(*vi*) Thus

$$\int \frac{x}{x^2+4x+8}dx = \frac{\ln(x^2+4x+8)}{2} - \arctan(\frac{x+2}{2}) + C$$

Example 4.3. Evaluate

$$\int \frac{x}{(x^2+2x+2)(x-1)}dx$$

First note that $b^2 - 4ac = 4 - 8 = -4 < 0$, so $x^2 + 2x + 2$ is irreducible as a quadratic function. Therefore, we need to apply the method of partial fractions:

$$\frac{x}{(x^2+2x+2)(x-1)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-1}$$

giving $x = (Ax+B)(x-1)+C(x^2+2x+2)$. Substituting $x = 1$, we get $1 = 5C$ or $C = 1/5$. Substituting $x = 0$, we get $0 = -B + 2/5$ or $B = 2/5$. Finally, substituting $x = -1$, we get $-1 = (-A + 2/5)(-2) + 1/5$ or $2A = -1 - 1/5 + 4/5 = 2/5$, so $A = 1/5$. Thus we have

$$\frac{x}{(x^2+2x+2)(x-1)} = \frac{x+2}{5(x^2+2x+2)} + \frac{1}{5(x-1)}.$$

In order to integrate, we need to use some trigonometric identities and the method of completing the square.
First observe that the last integral is a simple logarithmic substitution (the integral being $(\ln(x-1))/5$), so we only concern ourselves with the last integral. In this case, we proceed as follows:

(i) We complete the square of the denominator: $x^2 + 2x + 2 = (x+1)^2 + 1$.

(ii) Next we make a u substitution for the linear term resulting from completing the square: $u = x + 1$, so $du = dx$ and $x = u - 1$, giving

$$\frac{1}{5}\int \frac{x+2}{(x^2+2x+2)}dx = \int \frac{u+1}{u^2+1}du$$
$$= \frac{1}{5}\left[\int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du\right]$$

(iii) Now observe that the first integral is a simple logarithmic substitution and the second is an inverse tangent:

$$\frac{1}{5}\left[\frac{\ln(u^2+1)}{2} + \arctan(u)\right] = \frac{\ln(x^2+2x+2)}{10} + \frac{\arctan(x+1)}{5} + C$$

Thus

$$\int \frac{x}{(x^2+2x+2)(x-1)}dx$$
$$= \frac{\ln(x^2+2x+2)}{10} + \frac{\arctan(x+1)}{5} + \frac{\ln(x-1)}{5} + C$$

5. $q(x)$ HAS QUADRATIC POWERS WHICH MAY BE REPEATED

In this case, if $(ax^2+bx+c)^r$ is a factor of $q(x)$ where ax^2+bx+c is irreducible, then the terms

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

appear in the partial fraction decomposition of $R(x)$. The calculations are exactly the same as the previous cases, so to illustrate, we show one easy example.

Example 5.1. Evaluate the integral

$$\int \frac{x^3}{(x^2+1)^2} dx.$$

For this, we need to use the method of partial fractions. We observe that

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+1}.$$

Clearing denominators, we get $x^3 = (Ax+B)(x^2+1) + (Cx+D)$. From this point we need to solve for A, B, C and D by evaluating the functions at different points. Plugging in $x = 0$, we get $B = -D$. Plugging in $x = 1$, we get $1 = 2(A+B) + (C-B)$, for $x = -1$ we get $-1 = 2(-A+B) + (-C-B)$ and plugging in $x = 2$, we get $8 = 5(2A+B) + (2C-B)$. Solving these equations, we get $A = 1$, $B = 0$, $C = -1$ and $D = 0$, so

$$\frac{x^3}{(x^2+1)^2} = \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}.$$

Integrating, we get

$$\begin{aligned}\int \frac{x^3}{(x^2+1)^2} dx &= \int (\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}) dx \\ &= \frac{\ln(x^2+1)}{2} + \frac{1}{2(x^2+1)} + C\end{aligned}$$

This covers all possible scenarios for partial fractions. Remember, the method is the most important aspect of partial fractions - the point being that in principle any rational function can be integrated - though in practicality outside of homework, the problems I give you will be fairly straight forward. We finish with a different type of application for partial fractions.

6. Functions which are almost Rational

Similar methods to the ones we have used for rational functions can also be used for functions which are quotients of other functions though not necessarily rational (so not a quotient of polynomials). We illustrate by example.

Example 6.1. Evaluate

$$\int \frac{1}{x\sqrt{(x+1)}} dx.$$

Here we make the substitution $u = \sqrt{(x+1)}$, so $u^2 = x+1$ giving $x = u^2 - 1$ and $2udu = dx$. Evaluating, we get

$$\int \frac{1}{x\sqrt{(x+1)}} dx = \int \frac{1}{(u^2-1)u} 2udu = 2\int \frac{1}{(1-u)(1+u)} du.$$

Using partial fractions, we get $1/(u-1)(u+1) = A/(u-1)+B/(u+1)$, so $1 = A(u+1)+B(u-1)$. Making the substitution $u = 1$ gives $A = 1/2$ and $u = -1$ gives $B = -1/2$. Thus

$$\int \frac{1}{x\sqrt{(x+1)}}dx = 2\int \frac{1}{(u-1)(u+1)}du$$

$$= \int \frac{1}{u-1} - \frac{1}{u+1}du = \ln(u-1) - \ln(u+1) + C$$

$$= \ln(\frac{(\sqrt{(x+1)}-1)}{(\sqrt{(x+1)+1})}) + C$$