Section 7.5: Strategies for Integration

1. Which Method to Use

Unlike differentiation, there are no clear cut rules to evaluate every type of integral - indeed, there are some integrals which do not have algebraic antiderivatives. This makes the method of integration that much more challenging than that of differentiation. To try to alleviate some of the difficulties which can be encountered when trying to evaluate an integral, we outline an approach which when taken, will often lead to a solution.

(i) **Is the integral a standard integral or a modified version of a standard integral?**

There are many functions of which we should just know the integral: examples include $1/(x^2 + a^2)$ or $1/x$. Always check first that the integral is not one we should already know (you will have tables on the tests).

Example 1.1. Evaluate

$$\int \frac{4}{3x} dx.$$

This looks much more complicated than it really is - it is just a modified derivative of a logarithm, so we have

$$\int \frac{4}{3x} dx = \frac{4}{3} \int \frac{1}{x} dx = \frac{4}{3} \ln(|x|) + C.$$

(ii) **Simplify the integrand if possible**

There are many functions which seem too complicated to integrate, but after fairly straight forward simplification, they may be much easier.

Example 1.2. Evaluate

$$\int \frac{x+1}{x^2 - 1} dx.$$

This question looks like a partial fraction question, but after simplification, we get

$$\int \frac{x+1}{x^2 - 1} dx = \int \frac{x+1}{(x-1)(x+1)} dx = \int \frac{1}{(x-1)} dx = \ln(|x-1|) + C$$

provided $x \neq -1$.

(iii) **Attempt straight forward subsitution**

Before trying any of the more technical approaches we have discussed, check whether an easy substitution will work - it will save time even in circumstances where other more complicated methods will work.

Example 1.3. Evaluate

$$\int x\sqrt{x-1}dx.$$

This looks like a complicated integral which could be done using integration by parts or perhaps a carefully chosen trig substitution. However, a simple substitution is all that is required. Specifically, if we let $u = x - 1$, so $x = u + 1$ and $dx = du$, then we have

$$\int x\sqrt{x-1}dx = \int (u+1)\sqrt{u}du = \int (u^{\frac{3}{2}} + u^{\frac{1}{2}})du$$

$$= \frac{2u^{\frac{5}{2}}}{5} + \frac{3u^{\frac{3}{2}}}{2} + C = \frac{2(x-1)^{\frac{5}{2}}}{5} + \frac{2(x-1)^{\frac{3}{2}}}{3} + C.$$

(iv) **Classify the integral according to its form**

In this chapter, we considered a number of different types of integration methods depending upon the form of the integral. If the previous three steps have not allowed us to evaluate the integral, we can look at its type, and then apply one of the techniques we have developed. Specifically, we have:

(a) Trigonometric Functions: Attempt the reduction techniques

(b) Rational Functions: Use partial fractions

(c) Integration by parts: If it is a product, attempt integration by parts

(d) Radicals: If the integrand involves radicals, consider using trigonometric substitution

(v) **Keep Trying**

Just because the methods you have used do not immediately work does not mean that an antiderivative does not exist. In addition to using the integration techniques we have developed, there are a number of other useful tools you may use:

(a) Keep trying substitution or parts, perhaps choosing the functions in a way which may not seem to make sense. As in Example 1.3, a non standard choice could lead to a correct solution. Also, when all is said and done, barring algebraic simplification, substitution and parts are the only rules we really have.

(b) Use algebra and identities to rewrite the integral in equivalent forms.

Example 1.4. Evaluate

$$\int \sin^2(x) + \cos^2(x)dx.$$

This looks like a complicated problem, but after using the identity $\sin^2(x) + \cos^2(x) = 1$, we get

$$\int \sin^2(x) + \cos^2(x)dx = x + C.$$

(vi) You may have to use more than one method to evaluate an integral.

We illustrate with a couple of explicit examples.

Example 1.5. Evaluate

$$\int x \csc(x) \sec(x)dx.$$

We use integration by parts to evaluate this. Specifically, we choose $u = x$, $dv = \csc(x)\cot(x)$ and then we get $du = 1$ and $v = -\csc(x)$, so

$$\int x \csc(x) \sec(x)dx = -x \csc(x) - \int -\csc(x)dx$$

$$= -x \csc(x) + \ln|\csc(x) - \cot(x)|.$$

Example 1.6. Evaluate

$$\int \frac{1}{x + 4 + 4\sqrt{x + 1}}dx.$$

First we use substitution with $u = \sqrt{x + 1}$, so $x = u^2 - 1$. Then simplifying, we have

$$\int \frac{1}{x + 4 + 4\sqrt{x + 1}}dx = \int \frac{1}{u^2 + 4u + 3}du = \int \frac{1}{(u + 1)(u + 3)}du.$$

Here we can use partial fractions. Specifically, we have

$$\int \frac{1}{x + 4 + 4\sqrt{x + 1}}dx = \int \frac{1}{(u + 1)(u + 3)}du$$

$$= \int \left(-\frac{1}{u + 1} + \frac{3}{u + 3} \right)du = -\ln(+1) + 3\ln(u + 3) + C$$

$$= -\ln(\sqrt{x + 1} + 1) + 3\ln(\sqrt{x + 1} + 3) + C$$