Section 7.8: Improper Integrals
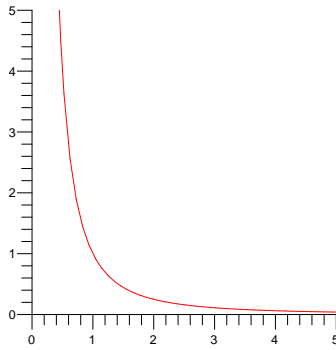
"The Infinite is Wierd"

1. The Definition of an Improper Integral

An improper integral is a definite integral of a function $f(x)$ in which either the limits are infinite or function $f(x)$ has an asymptote over the region of integration. Even though it seems that such integrals should always be infinite, there are some very interesting results which show they are not always as such. We shall look at the two different types of improper integrals seperately motivating each with examples.

2. Infinite Intervals

Suppose we want to determine the area bounded between $f(x) = 1/x^2$ and the x-axis for $x \geqslant 1$.

Since the interval over which we are trying to find the area is infinite, it seems as if the area should be infinite. However, the area is also getting very small very quickly as x gets large. So question we really need to ask is "Is the area getting smaller quickly enough that at some point, the amount of additional area is negligible?

In order to determine this, we start by finding the definite integral of $f(x)$ over large regions starting at 1. Suppose that t is some very large number. Then

$$\int_1^t \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^t = 1 - \frac{1}{t}.$$

Observe that as we take the integral over larger and larger values of t, the value of the integral gets closer and closer to 1. Therefore, it seems that the area bounded between the x-axis and $f(x)$ for $x \geqslant 1$ is 1. That is

$$\int_1^\infty \frac{1}{x^2} dx = 1.$$

With this in mind, the natural question to ask, which we have basically answered, is:

Question 2.1. How can we define a definite integral over an infinite region?

Result 2.2.   (*i*) If $f(x)$ is continuous for all $x \geqslant a$, and $\int_a^t f(x)dx$ exists for all $t \geqslant a$, we define

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(*ii*) If $f(x)$ is continuous for all $x \leqslant b$, and $\int_t^b f(x)dx$ exists for all $t \leqslant b$, we define

$$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

(*iii*) If $f(x)$ is continuous for all $x \geqslant a$, and all $x \leqslant a$, and $\int_a^t f(x)dx$ exists for all $t \geqslant a$ and $\int_t^a f(x)dx$ exists for all $t \leqslant a$ then we define

$$\int_{-\infty}^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx + \lim_{t \to -\infty} \int_t^a f(x)dx$$

provided these limits exists (as finite numbers).

The improper integrals $\int_a^\infty f(x)dx$, $\int_{-\infty}^b f(x)dx$, $\int_{-\infty}^\infty f(x)dx$ are called convergent if the corresponding limits exist and divergent if the limits do not exist.

We look at some examples of how to evaluate improper integrals.

Example 2.3.   (*i*) Evaluate $\int_1^\infty \frac{\ln(x)}{x^3}dx$.

$$\int_1^\infty \frac{\ln(x)}{x^3}dx = \lim_{t \to \infty} \int_1^t \frac{\ln(x)}{x^3}dx$$

so taking $dv = 1/x^3$, $u = \ln(x)$, we get $v = -1/2x^2$ and $du = 1/x$, so

$$\lim_{t \to \infty} \int_1^t \frac{\ln(x)}{x^3}dx = \lim_{t \to \infty} \left[-\frac{\ln(x)}{2x^2} + \int \frac{1}{2x^3}dx \right]_1^t$$

$$= \lim_{t \to \infty} \left[-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} \right]_1^t = \lim_{t \to \infty} (-\frac{\ln(t)}{2t^2} - \frac{1}{4t^2}) - (-\frac{1}{4}) = \frac{1}{4}$$

(*ii*) Evaluate $\int_{-\infty}^\infty \frac{x}{1+x^2}dx$.

We break the integral up into two different integrals with midpoint 0. Using substitution with $u = x^2 + 1$ so $du/2x = dx$, we get

$$\int_0^\infty \frac{x}{x^2+1}dx = \int_0^\infty \frac{1}{2u}du = \lim_{t \to \infty} \frac{\ln(x^2+1)}{2} \Big|_0^t$$

and this limit is undefined. In particular, the integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx$ diverges.

3. ASYMPTOTES IN INTERVALS

The second form of indefinite integral is when there is an asymptote somewhere in the region of integration. Just as with the previous case, it seems that any integral over a region with an asymptote should be infinite. However, as with the previous case, if the area being added gets small enough quickly enough, then the integral might be finite. Before we look at any examples, we start with a formal definition and approach toward the calculation of these limits.

Result 3.1. (i) If $f(x)$ is continuous on $[a, b)$, then we define

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).
(ii) If $f(x)$ is continuous on $(a, b]$, then we define

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).
(iii) If $f(x)$ is continuous on $[a, b]$ except at $c \in [a, b]$, then we define

$$\int_a^b f(x)dx = \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$$

provided these limits exists (as finite numbers).
These integrals are called convergent if the corresponding limits exist and divergent if the limits do not exist.

We look at some examples of how to evaluate improper integrals.

Example 3.2. (i) Evaluate $\int_1^9 \frac{1}{(x-9)^{1/3}}dx$.
For this equation, we observe that there is an asymptote at $x = 9$, so we need to consider the limit

$$\lim_{t \to 9^-} \int_1^t \frac{1}{(x-9)^{1/3}}dx = \lim_{t \to 9^-} \frac{3}{2}(x-9)^{2/3}\Big|_1^t = \lim_{t \to 9} \frac{3}{2}((-8)^{2/3}-(t-9)^{2/3}) = 6$$

(ii) Evaluate $\int_0^4 \frac{1}{x^2+x-6}dx$.
First observe that $x^2 + x - 6 = (x + 3)(x - 2)$, so there is an asymptote at $x = 2$. In order to integrate this function, we need to use the method of partial fractions:

$$\frac{1}{(x + 3)(x - 2)} = \frac{A}{(x - 2)} + \frac{B}{(x + 3)}$$

or $1 = A(x + 3) + B(x - 2)$. Plugging in $x = 2$, we get $5B = 1$ or $B = 1/5$. Plugging in $x = -3$, we get $1 = -5A$

or $A = -1/5$. Both integrals are standard logarithmic type integrals. Integrating, we get:

$$\int_0^4 \frac{1}{x^2 + x - 6}dx$$

$$= \lim_{t \to 2^-} \int_0^t -\frac{1}{5(x-2)} + \frac{1}{5(x+3)}dx + \lim_{t \to 2^+} \int_t^4 -\frac{1}{5(x-2)} + \frac{1}{5(x+3)}dx$$

$$= \lim_{t \to 2^-} \left[\frac{\ln|x-2|}{5} + \frac{\ln|x+3|}{5}\right]_0^t + \lim_{t \to 2^+} \left[\frac{\ln|x-2|}{5} + \frac{\ln|x+3|}{5}\right]_t^4.$$

Observe that in both cases these limits do not exist and hence this integral is divergent.