Arc Length

In this section, we return to the idea of Riemann sums to answer the following question: given a curve C in the plane, how do we find the length of C? The applications of being able to answer such questions are very important in engineering, physics and the sciences.

1. Calculating Arc Length Using Riemann Sums

Our approach is as follows

(i) Suppose that C is a curve in the plane and assume that C is the graph of some function $f(x)$ on an interval $[a, b]$.

(ii) If C is curved, we cannot find the length of C directly. However, if C is a straight line, it is easy to find the length of the curve using pythagoras i.e. if C is a line with equation $y = mx + c$, then the length of C is equal to

$$\sqrt{((b-a)^2 + (m(b-a))^2)} = (b-a)\sqrt{(1+m^2)}.$$

(iii) In general, we can break up the interval $[a, b]$ into n smaller equally sized subintervals of lengths Δx. Let $x_0, \ldots, x_n$ be the end points of these intervals.

(iv) On the first interval, the arc length can be approximated by using the tangent line at x_1. Thus we will get Length

$$= (x_1 - x_0)\sqrt{(1 + (f'(x_0))^2)} = \sqrt{(1 + (f'(x_0))^2)}\Delta x.$$

(v) In general, in the ith subinterval, we can estimate the length of the curve using the tangent line at x_i giving

$$\sqrt{(1 + (f'(x_i))^2)}\Delta x.$$

(vi) To get the total length of C, we add up the length of each of the smaller subintervals:

$$\text{Length } = \sum_{i=1}^{n} \sqrt{(1 + (f'(x_i))^2)}.$$

(vii) This is a Riemann sum, so letting $\Delta x \to 0$, the sum becomes an integral. Thus we get the following:

Result 1.1. If $f(x)$ is a continuous function on $[a, b]$, then the arc length of the graph of $f(x)$ between $[a, b]$ is

$$\int_a^b \sqrt{(1 + f(x)^2)}dx.$$

We look at a number of examples.

Example 1.2. Find the length of the curve

$$y^2 = 4(x+4)^3$$

where $0 \leqslant x \leqslant 2$ and $y > 0$.

First we observe that $y = \pm 2(x+4)^{3/2}$ and since $y > 0$, we must have $y = 2(x+4)^{3/2}$ and $dy/dx = 3(x+4)^{1/2}$. Applying the arc length formula, we get

$$\text{Length } = \int_0^2 \sqrt{(1 + (3(x+4)^{1/2})^2)} = \int_0^2 \sqrt{(1 + 9(x+4))}.$$

Making the substitution $u = 9x + 37$, we have $(1/9)du = dx$ and $x = 0$ implies $u = 37$ and $x = 2$ implies $y = 55$, so

$$= \int_{37}^{55} \sqrt{u}\frac{1}{9}du = \frac{2}{27}u^{3/2}\Big|_{37}^{55} = \frac{2}{27}(55^{(3/2)} - 37^{(3/2)})$$

As with all other things we have considered, we can also consider functions of y instead of x and the formula still holds.

Example 1.3. Find the arc length of $y^2 = 4x$ or $x = y^2/4$ for $0 \leqslant y \leqslant 2$.

First observe that $dx/dy = y/2$, so applying the formula directly, we get

$$\int_0^2 \sqrt{1 + \frac{y^2}{4}}dy = \frac{1}{2}\int_0^2 \sqrt{4 + y^2}dy.$$

Trigonometric substitution does not work in this case, so we use tables giving

$$\frac{1}{2}\int \sqrt{4+y^2}dy = \frac{1}{2}\left[\frac{y}{2}\sqrt{4+y^2} + 2\ln(y + \sqrt{4+y^2})\right]_0^2$$
$$= \frac{1}{2}\left[\sqrt{8} + 2\ln(2 + \sqrt{8}) - 2\ln(\sqrt{4})\right]$$

2. Applications - The Arc Length Function

Instead of trying to find a specific arc length, sometimes when we are given a function, we would like to be able to find a formula for arc length which depends upon one of the endpoints of the interval. In this case, the endpoint would be a variable, and the integral would be a function whose value at any point is the length of the arc over that interval. All calculations would be the same, so we would have:

Result 2.1. Suppose $y = f(x)$ is a continuous function on $[a, b]$ and x is a number in the interval $[a, b]$. Then the arc length from a to x is a function $s(x)$ of x given by the equation

$$s(x) = \int_a^x \sqrt{(1 + [f'(t)]^2)}dt.$$

Example 2.2. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $(1, 2)$.
Let $f(t) = 2x^{3/2}$. Then $f'(t) = 3x^{1/2}$, so

$$\sqrt{(1 + [f'(t)]^2)} = \sqrt{(1 + [3t^{(1/2)}]^2)} = \sqrt{(1 + 9t)}.$$

Then

$$\int_1^x \sqrt{(1 + 9t)}dt = \frac{2}{27}(1 + 9t)^{3/2}\bigg|_1^x = \frac{2}{27}\left[(1 + 9x)^{3/2} - (10)^{3/2}\right].$$

This means rather than calculating the arc length at each different point separately, we now have a formula for arc length from 1 to any value x. Thus, the arc length to 4 would be $\frac{2}{27}\left[(19)^{3/2} - (10)^{3/2}\right]$ and the arc length to 10 would be $\frac{2}{27}\left[(91)^{3/2} - (10)^{3/2}\right]$.