Section 9.3: Separable Equations

A differential equation is an equation which involves an unknown function and some of its derivatives. To solve a differential equation is to find a function which satisfies it. Differential equations are probably the most important applicable area of mathematics with applications in physics, engineering and chemistry, to name but a few. Though a thorough study of Differential equations is a complete course in its own right, we shall look at a special class of differential equations which we will be able to solve.

1. Separable Differential Equations

We start with the definition of a separable differential equation.

Definition 1.1. A separable equation is a first order differential equation in which the expression for dy/dx can be factcored as a function of x times a function of y. In other words, it is an equation of the from

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$

(we write it as a fraction for convenience).

To solve this type of equation, we rewrite as $f(y)dy = g(x)dx$, and then we can integrate both sides:

$$\int f(y)dy = \int g(x)dx$$

and we can solve for y. Of course, we should check that this really will solve the system, (that is, the equation will be satisfied). In order to do this, we need to differentiate both sides with respect to x to check:

$$\frac{d}{dx}\int f(y)dy = \frac{d}{dx}\int g(x)dx.$$

Using implicit differentiation and the fundamental theorem of Calculus, we get

$$\frac{d}{dx}\int f(y)dy = f(y)\frac{dy}{dx} = g(x),$$

so the differential equation is satisfied.
We start by looking at some example to help recognize separable equations.

Example 1.2. Which of the following are separable differential equations:

(*i*) $y'x + x = 1$

Solving for y', we get

$$y' = \frac{1-x}{x},$$

so this is a separable equation.

(*ii*) $y'x = 1$

Solving for y', we get

$$y' = \frac{1}{x},$$

so this is a separable equation.

(*iii*) $x + y = y'$

This equation is already solved for y'. Note that there is no way we can write it in the form $y' = f(y)/g(x)$, and thus it is not a separable equation.

(*iv*) $y' = \ln(x) + \ln(y)$

Similar to the last problem, this equation is already solved for y'. Note that there is no way we can write it in the form $y' = f(y)/g(x)$, and thus it is not a separable equation.

We now consider problems to show how to solve separable equations.

Example 1.3. Solve the following separable differential equations.

(*i*) $dy/dx = y/x$.

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx,$$

so

$$\ln(y) = \ln(x) + C$$

or

$$y = e^{\ln(x)+C} = Ax$$

where $A = e^C$.

(*ii*) $y' = xy/2\ln(y)$.

Here we have

$$\int \frac{2\ln(y)}{y} dy = \int x dx.$$

Using substitution on the left hand side with $u = \ln(y)$, we get

$$(\ln(y))^2 = x^2/2 + C$$

or

$$\ln(y) = \pm\sqrt{x/2 + C}$$

giving

$$y = e^{\pm\sqrt{x/2+C}}.$$

(*iii*) $dy/dx = y^2 + 1$ with $y(1) = 0$.

Here we have

$$\int \frac{1}{y^2+1} dy = \int dx$$

so $\arctan(y) = x + C$ or $y = \tan(x + C)$. Substituting $x = 1$, we have $0 = tan(1 + C)$ or $C = -1$, so $y = \tan(x - 1)$.

(*iv*) $xy' + y = y^2$.

Before we solve this equation, we need to rewrite it in separable form. Observe that $xy' = y^2 - y$, so $y' = (y^2 - y)/x$ giving,

$$\int \frac{1}{y^2-y} dy = \int \frac{1}{x} dx.$$

We need to use partial fractions on the first integral. Observe that

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

so $1 = A(y - 1) + By$. This gives $A = -1$ and $B = 1$, so

$$\int \frac{1}{y^2-y} dy = \int -\frac{1}{y} + \frac{1}{y-1} dy = -\ln(y) + \ln(y-1) = \int \frac{1}{x} dx = \ln(x) + C.$$

Using logarithm properties, we get

$$-\ln(y) + \ln(y-1) = \ln((y-1)/y) = \ln(x) + C$$

so

$$(y-1)/y = Kx$$

where $K = e^C$. Thus $1 - 1/y = Kx$, or $1/y = 1 - Kx$ giving

$$y = \frac{1}{1-Kx}.$$

Other examples follow a similar pattern.