Review

"If you don't know this, REVIEW!".

The following is a general review of Precalculus, Calculus I, and II with an emphasis on some of the topics which will be very important in Vector Calculus Multivariable Calculus. If you do not recognize something or you have forgotten some of the techniques, **REVIEW NOW** - do not take the ostrich approach as it will be much harder to play catch up later in the semester. We break the review up into three pieces: precalculus, calculus and geometric techniques.

1. PreCalculus

Of course, all of precalculus is prerequisite to any calculus course. However there are some topics which are especially important which you may have forgotten.

(*i*) ALGEBRA, ALGEBRA, ALGEBRA

A good background in algebra is required in any discipline in which you manipulate numbers. If you are not strong at algebra, this is the time it will REALLY start to hurt - if need be, you should brush up on basic techniques in arithmetic (fractions, multiplication, division, powers, roots etc.) and the basic formulas (quadratic formula, distance formula etc.)

(*ii*) Geometry

Basic techniques in geometry, plotting points in the plane, in space etc.

(*iii*) Functions

Functions are a key concept in all of mathematics. You should know what a function is and all of the buzz words associated with a function like the following: domain, range, codomain, graph, 1-1, onto, continuous, discrete etc.

(*iv*) Working with Variables

There is a reason this course is called Multivariable Calculus - get used to equations involving two or more variables!

(*v*) More Complicated Algebraic Techniques

Though the emphasis in Vector Calculus is understanding and not simply computation, you will still be expected to employ certain techniques from precalculus and calculus to solve problems. Though not an exhaustive list, these skills include: completing the square of a quadratic function, partial fractions, trigonometric identities to simplify trigonometric functions.

We illustrate with an example.

Example 1.1. Complete the square of both variables in the following equation:

$$x^2 + 2x + y^2 - 4y - 2 = 8$$

Completing the square in x, we have

$$x^2 + 2x = (x + 1)^2 - 1$$

and completing the square in y, we have

$$y^2 + 4y = (y - 2)^2 - 4.$$

Substituting back into the equation, we have

$$x^2 + 2x + y^2 - 4y - 2 = (x + 1)^2 - 1 + (y - 2)^2 - 4 - 2 = 8.$$

Simplifying, we have

$$(x + 1)^2 + (y - 2)^2 = 15.$$

2. Calculus

Calculus is the accumulation of techniques developed in Calculus I and Calculus II - being able to differentiate and integrate many different types of functions. However, in addition to the topics of integration and differentiation of functions, there are other techniques involving differentiation and integration developed in Calculus I and II which will be relevant to this course. The following is a list of the things I expect you to know broken up into differentiation topics and integration topics.

- (i) **Working with lots of variables**

  This class is sometimes called multivariable calculus for a reason - you will be exploring properties of functions and equations which involve lots of different variables. During your algebra and calculus classes, you probably looked at many different problems which involved more than one variable in an expression - in Vector Calculus, this will be absolutely standard, so you should get comfortable working with such equations.
- (ii) **Limits**
  - (a) The definition of a limit
  - (b) Basic rules of limits
  - (c) More sophisticated rules (L'Hopitals rule, derivatives as limits)
- (iii) **Differentiation**
  - (a) Rules of Differentiation
    - (i) Basic Rules (linearity, constant multiple etc.).
    - (ii) Basic Functions
    - (iii) The product rule (products).

(iv) The quotient rule (quotients).
(v) The chain rule (compositions).

Example 2.1. Differentiate the following functions:
(i) $f(x) = xe^{x^2}$

We have
$$f'(x) = e^{x^2} + x(2xe^{x^2}) = e^{x^2}(1 + 2x^2).$$

(ii) $f(x) = \frac{\sin(x)-1}{x^2}$

We have
$$f'(x) = \frac{\cos(x)x^2 - 2x(\sin(x) - 1)}{x^4} = \frac{x^2\cos(x) - 2x\sin(x) + 2x}{x^4}.$$

(b) Methods of Differentiation
(i) Implicit Differentiation.
(ii) Logarithmic Differentiation.

Example 2.2. Differentiate the following functions with respect to x:
(A) $xy = x^2$

We have
$$y + xy' = 2x \text{ so } xy' = 2x - y$$
and thus
$$y' = \frac{2x - y}{x}.$$

(B) $f(x) = x^x$

Setting $y = x^x$ and taking the natural logarithm of both sides, we have
$$\ln(y) = \ln(x^x) = x\ln(x).$$
Differentiating implicitly, we get
$$\frac{1}{y}y' = \ln(x) + x\frac{1}{x} = \ln(x) + 1.$$
Solving for y', we get
$$y' = x^x(\ln(x) + 1).$$

(c) Linear Approximations
Perhaps one of the most important facts in Calculus I is that any "smooth" function can be approximated by a straight line - its tangent line or linear approximation. This is important because it means that even if a function is extremely complicated, provided it is smooth, we can always approximate its values through the use of a tangent line, and this is good because LINES ARE EASY

TO UNDERSTAND!!! In multivariable calculus, we shall generalize this idea to functions with many variables.

Example 2.3. Find the tangent line to $f(x) = e^x$ at $x = 0$.

We need a point and a slope to write down the equation for the tangent line. The point is $(0, e^0) = (0, 1)$. We use the derivative to find the slope: $f'(x) = e^x$, so the slope at $x = 0$ is $f'(0) = e^0 = 1$. Thus the tangent line has equation

$$l(x) = 1(x - 0) + 1 = x + 1.$$

Example 2.4. Find the linear approximation to $f(x) = \ln(x)$ at $x = 1$ and use it to estimate $\ln(1.1)$.

The linear approximation is simply the tangent line. We need a point and a slope to write down the equation for the tangent line. The point is $(1, \ln(1)) = (1, 0)$. We use the derivative to find the slope: $f'(x) = 1/x$, so the slope at $x = 1$ is $f'(0) = 1$. Thus the tangent line has equation

$$l(x) = 1(x - 1) + 0 = x - 1.$$

The tangent line approximates $\ln(x)$ close to $x = 1$, so

$$\ln(1.1) \simeq l(1.1) = 1.1 - 1 = 0.1$$

(note that the calculator gives 0.95, so the linear approximation is quite good).

(iv) Integration
- (a) Integration of Basic Functions
- (b) Integration techniques:
  - (i) Integration by Substitution
  - (ii) Integration by Parts
  - (iii) Trigonometric Integrals
  - (iv) Trigonometric Substitution
  - (v) Partial Fractions
  - (vi) Approximate Integration (Simpsons Rule, Midpoint, Trapezoid)
  - (vii) Improper Integrals

Example 2.5. Evaluate the following:

(i)

$$\int \frac{2}{x^2 - 1} dx$$

The function $1/(x^2+1)$ is the derivative of $\arctan(x)$, so using the constant multiple rule, we have

$$\int \frac{2}{x^2+1}dx = 2\arctan(x) + C$$

(ii)

$$\int a\sin^2(x)dx$$

Using the half angle identities, we know

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

so we have

$$\int a\sin^2(x)dx = \int a\frac{1-\cos(2x)}{2}dx = \frac{a}{2}\int 1-\cos(2x)dx$$

$$= \frac{a}{2}\left(x - \frac{\sin(2x)}{2}\right) + C = \frac{ax}{2} - \frac{a\sin(2x)}{4} + C.$$

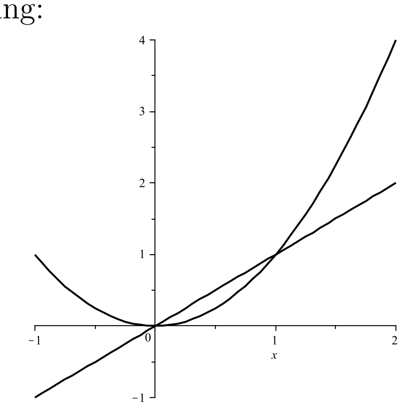
(c) The Fundamental Theorem of Calculus:
If F is an antiderivative of f then

$$\int_a^b f(x)dx = F(b) - F(a).$$

(d) Riemann Sums and the Definite Integral.
Recall that the definite integral was constructed using Riemann sums. Riemann sums are especially important in Multivariable calculus, so you need to know the basic definitions, and how to evaluate them.

Example 2.6. Find the area bounded between the curves $y = x^2$ and $y = x$ between using a Riemann sum. Then use calculus to get the exact value.

Sketching the graphs of these two functions, we get the following:



Clearly they intersect at $x = 0$ and $x = 1$, so the area bounded between them will be bounded between these two values of x. Using a Riemann sum with two subdivisions and using midpoints to estimate this area, we get

$$\text{Area} \simeq \frac{1}{2}\left(\left(\frac{1}{4} - \frac{1}{16}\right) + \left(\frac{3}{4} - \frac{9}{16}\right)\right) = \frac{3}{16}.$$

Using calculus, we can find the exact area:

$$\text{Area} = \int_0^1 (x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3. Geometric Techniques

In addition to precalculus and calculus, you will also need to recall geometric techniques developed in your previous calculus classes and your geometry classes. Some of the main techniques we need will be the following:

(*i*) Graphing Techniques in Space

Much of vector calculus will take place in 3-space, so it will be necessary to modify the techniques developed in elementary geometry in the plane to a three dimensional coordinate system. This will include basic problems such as plotting points, graphing functions and curves, finding distance between objects in space (generalizing the Pythagoras).

(*ii*) Polar Coordinates

Polar coordinates and other coordinate systems will be very important in multivariable calculus. They were introduced to make certain integrals and derivatives easier. When we get to the relevant sections, we will recap a little, but you should be aware of most of the basic ideas behind polar coordinates.

Example 3.1. Write down a polar function for a circle of radius 3 centered at the origin and use integration in polar coordinates to find its area, remember,

$$A = \int_a^b \frac{1}{2} r^2 d\vartheta$$

A polar function for a circle of radius 3 is $r = 3$ where $0 \leqslant \vartheta \leqslant 2\pi$. Then using the formula, we have

$$A = \int_0^{2\pi} \frac{1}{2} 9 d\vartheta = \frac{9}{2}\vartheta\bigg|_0^{2\pi} = 9\pi.$$

Note that this agrees with the usual formula for the area of a circle.

(*iii*) Parameterization of Curves

Parametrization will be crucial when trying to represent curves in three space (indeed, with parametric equations, it would not be possible). Loosely speaking, a parametrization of a curve is a set of functions for the coordinates of a moving object in terms of variable t (usually considered time). There are two very important techniques for parametric curves you should be comfortable with (the most important is the second):

(a) Given equations for a parametric curve, working with them.

Example 3.2. Sketch the curve C with parametric equations $x(t) = \cos(t)$, $y(t) = -\sin(t)$.

Since $x^2 + y^2 = 1$, these parametric equations trace out a circle of radius 1 centered at the origin. The initial position (when $t = 0$) is $(1, 0)$ and the orientation of the curve is clockwise since as t increases, $y(t)$ becomes negative.

(b) Deriving parametric equations for a curve.

Example 3.3. Find the equation of a circle of radius 2, centered at $(0, 1)$ oriented clockwise.

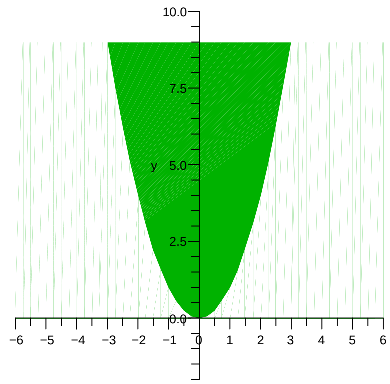
We have $x = \cos(x)$ and $y(t) = \sin(t)$ as the standard parametric equations for a circle of radius 1 centered at the origin starting at $(1, 0)$ and oriented counterclockwise. Therefore, the equations $x = 2\cos(x)$ and $y(t) = -2\sin(t) + 1$ are parametric equations for a circle of radius 2 centered at $(0, 1)$ oriented clockwise.

(*iv*) Bounding Regions

To bound a region means to write down a set of inequalities which represent the region you are trying to bound. This is perhaps something you have not considered since you took algerba. However, in multivariable calculus, it will be very important to learn how to bound regions in lots of different coordinate systems, so make sure to spend some time brushing up on how to do this. We illustrate with a couple of examples in different coordinate systems. Loosely speaking, we would use polar coordinates to bound areas made up from circles, and cartesian else (though we shall introduce more coordinate systems).

Example 3.4. Bound the following region in either polar coordinates or Cartesian coordinates.

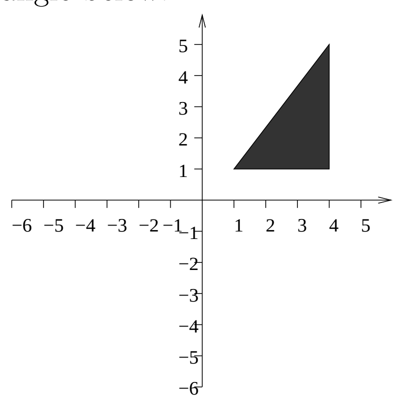
(a) The region above $y = x^2$ below the line $y = 9$.



In Cartesian coordinates, this would be the region

$$x^2 \leqslant y \leqslant 9$$
$$-3 \leqslant x \leqslant 3$$

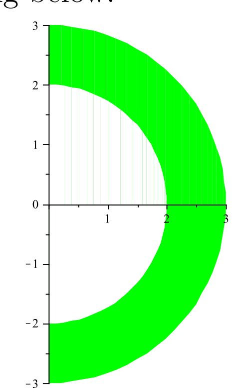
(b) The triangle below.



The line making up the hypotenuse of the triangle is $y = \frac{4}{3}(x-1)+1$ which is the upper bound for y. Therefore, in Cartesian coordinates, this would be the region

$$1 \leqslant y \leqslant \tfrac{4}{3}(x-1)+1$$
$$1 \leqslant x \leqslant 4$$

(c) The half ring below:



Since this is a circular region, polar coordinates would probably be best. The outer circle has radius 3, so polar equation $r = 3$, and the inner circle radius 2, so polar equation $r = 2$. The angle is bounded between $-\pi/2$ and $\pi/2$, so this region is bounded by

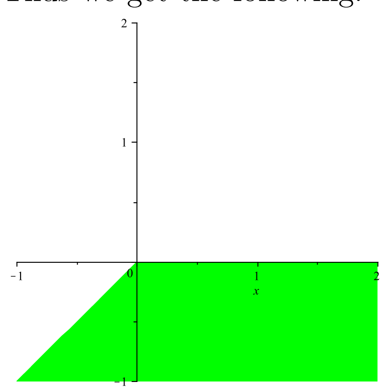
$$2 \leqslant r \leqslant 3$$
$$-\tfrac{\pi}{2} \leqslant \vartheta \leqslant \tfrac{\pi}{2}$$

Example 3.5. Sketch the following regions:

(a) The region bounded by

$$y \leqslant x \leqslant 1$$
$$-1 \leqslant y \leqslant 0$$

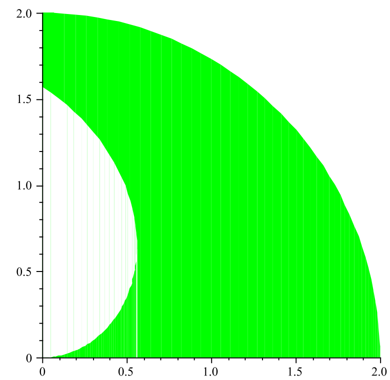
Since $x \geqslant y$, we know that the x values can never occur to the left hand side of the line $y = x$, so the line $y = x$ must bound this region. Next, since $x \leqslant 1$, no x values larger than 1 can occur, so the line $x = 1$ must bound this region. Finally, y must be between -1 and 1, so all values in the region we have already bound (between $y = x$ and $x = 1$), with y values between -1 and 1 will be in this region. Thus we get the following:



(b) The region bounded by

$$t \leqslant r \leqslant 2$$
$$0 \leqslant \vartheta \leqslant \tfrac{\pi}{2}$$

First, we sketch $r = t$ and $r = 2$, the two functions which bound the region. Next, we restrict to $0 \leqslant \vartheta \leqslant \pi/2$, and shade the corresponding region with the appropriate bounds on r. Thus we get the following:



2.0
1.5
1.0
0.5
0
0
0.5
1.0
1.5
2.0