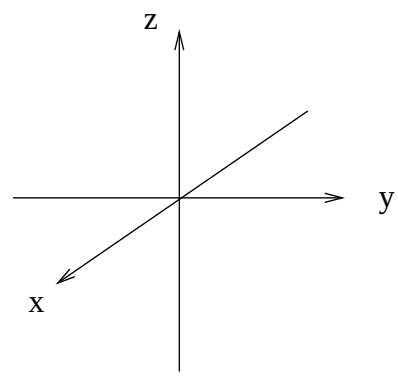
Section 13.1
Three-Dimensional Coordinate Systems

"Living in a 3-dimensional World"

1. Three-Dimensional Rectangular Coordinate System

We can sketch the graph of a function of two variables in the plane: the x-coordinate is the "input" value and the y coordinate is the corresponding output. To illustrate the graph of a function of two variables, we need two inputs and a single output - this means we need an extra dimension if we want to sketch a graph. We do this as following:

- (The right hand rule) With your right hand, point your thumb up, and your next finger and your middle finger outward perpendicular to each other. Draw three lines in the direction your thumb and fingers are pointing - label the line along your thumb the z-axis, your middle finger the y-axis and the remaining one the x-axis. The positive axis is in the direction your fingers point.



- Any point P in 3 space is completely determined by its distance along the x, y and z-axis. In order to sketch a point P in 3-space, we associate the ordered triple (a, b, c) where P is a directed distance of a units in the x-direction, b units in the y-direction, and c units in the z-direction. We will often denote a point P as $P(a, b, c)$ or (a, b, c).
- We call the three axis the **coordinate axis**.
- Any two axis determine a plane which we call a **coordinate plane**. They are referred to by the axis which determine them - the xy-plane, the yz-plane and the xz-plane.
- The three coordinate planes break up three space into eight parts called octants. The first octant is the octant where x, y and z are all positive.
- If we drop a perpendicular from any point $P(a, b, c)$ into one of the coordinate planes, we get a point in that plane called the

projection into that plane. We have $(0, b, c)$ as the projection into the yz-plane, we have $(a, 0, c)$ as the projection into the xz-plane, and $(a, b, 0)$ as the projection into the xy-plane.

Just as with two dimensions, any equations in 3-dimensions determine a graph in 3-space. Being able to recognize and sketch graphs in 3-space will be very important.

Example 1.1.
Describe and sketch the surface represented by $z = 2$.

This is all points with a z value of 2, so will be a horizontal plane at $z = 2$.

Describe and sketch the surface represented by $y = \sqrt{x}$.

At $z = 0$, the equation $y = \sqrt{x}$ will simply be the graph of $y = \sqrt{x}$ in the xy-plane. Since there are no conditions on z, we can extend this graph out vertically, and we the equation $y = \sqrt{x}$ will still be satisfied. Therefore, this will be the graph of $y = \sqrt{x}$ in the xy-plane extended vertically (in the z-direction) infinitely.

Describe and sketch the surface represented by $z = y$.

Similar to the previous example, at $x = 0$, the equation $z = y$ will simply be the graph of $z = y$ in the zy-plane. Since there are no conditions on x, we can extend this graph out in the x-direction, and we the equation $z = y$ will still be satisfied. Therefore, this will be the graph of $z = y$ in the zy-plane extended horizontally (in the x-direction) infinitely. In particular, it will be a plane.

Example 1.2.
If you are stood at $(3, 2, 1)$ and are looking at $(1, 2, 3)$, are you looking up or down?

Up since the z value at the point you are looking at is higher than the point you are stood at.

If you lift the xy-plane up so it has z-coordinate 1, what will be the equation for this surface?

The xy-plane has equation $z = 0$. If we move it up 1, then it will have equation $z = 1$.

Write down the equation for a surface which when you move in the positive x-direction, z grows exponentially, but z stays fixed in the y-direction.

An example of such an equation would be $z = e^x$.

2. The Distance Formula and the Equation for a Sphere

To find the distance between any two points in three space, we use a very similar formula to that in 2-space. The idea is to generalize Pythagoras theorem.

Result 2.1. The distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The distance formula is simple to apply to find the distance between points.

Example 2.2. (*i*) Find the shortest distance from point $P(3, 8, -2)$ to the xy-plane.

This will just be the distance between the projection of P onto the xy-plane and the point P.

(*ii*) Find the shortest distance from the point $(3, 8, -2)$ to the z-axis.

This will just be the absolute value of the z coordinate.

A sphere of radius R centered at the point (a, b, c) is by definition the set of all points a distance R from the point (a, b, c) in 3-space. We can use the distance formula to determine a formula for such a sphere.

Result 2.3. An equation of a sphere with center (a, b, c) and radius R is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Example 2.4. Show that the graph of the equation $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$ is a sphere and find its radius and center.

To show it is a sphere, we complete the square in all three variables:

$$x^2 + y^2 + z^2 - 6x + 4y - 2z = (x - 3)^2 - 9 + (y + 2)^2 - 4 + (z - 1)^2 - 1 = 11$$

so

$$(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 25.$$

Thus it is a sphere of radius 5 and center $(3, -2, 1)$.

3. Regions in 3-Space

We know how to bound regions in 2-space. Being able to bound regions in 3-space is also important. We illustrate with a couple of examples.

Example 3.1. (*i*) Sketch the region represented by the inequalities $x^2 + y^2 + z^2 \leqslant 1$, $x \geqslant 1/2$.

This is the "scalp" of a sphere pointing in the x-direction.

(*ii*) Find bounds for the region which consists of a hollow ball with outer radius 5 and inner radius 4 centered at $(1, 2, 3)$.

Outer ball equation is $(x-1)^2+(y-2)^2+(z-3)^2 = 25$ and the inner ball equation is $(x-1)^2+(y-2)^2+(z-3)^2 = 16$. We want it to be smaller than the outer ball and larger than the inner ball, so we get $(x-1)^2+(y-2)^2+(z-3)^2 \leqslant 25$ and $(x-1)^2+(y-2)^2+(z-3)^2 \geqslant 16$