Section 13.2
Vectors

"Direction and Magnitude"

1. Basic Definitions

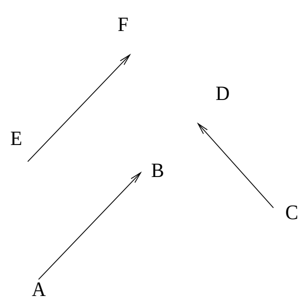
Vectors are one of the main ideas used in multivariable calculus. A vector is a quantity which consists of a **direction** and a **magnitude**. The following are all examples:

Velocity: direction is which way you are going and magnitude is speed.
Force: direction is which was the force is being directed, magnitude is the amount of force
Displacement: direction and distance of something moved from one point to another

We shall initially deal with displacement vectors because they are easy to understand - however, all the results we shall consider are true of general vectors. There are a number of important definitions and some new terminology we need.

(i) The displacement vector from point A to point B is an arrow with its tail at A and its tip at B.
(ii) We call A the initial point and point B the terminal point and denote the vector by $\vec{AB}$.
(iii) The magnitude of the vector $\vec{AB}$ is defined to be the length and the direction is the direction in which it points. We denote its magnitude by $||\vec{AB}||$.
(iv) Two vectors are considered equivalent or equal if they have the same magnitude and direction (even if they have different initial and end positions).
(v) The zero vector denotes $\vec{0}$ is defined to be the vector of magnitude 0. We define it to have no direction.

For example, in the picture below, $\vec{AB} = \vec{EF}$ but neither are equal to $\vec{CD}$.



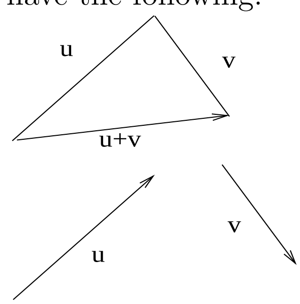
We note that when representing a vector by a letter, we shall always include a bar or arrow over it to avoid confusion with scalars (numbers) which are just magnitudes - the book does not do this!!! (So u is a number, but $\vec{u}$ or $\bar{u}$ is a vector).

2. Operations on Vectors

Just like with numbers, we can define basic arithmetic operations on vectors. There are three basic operations:

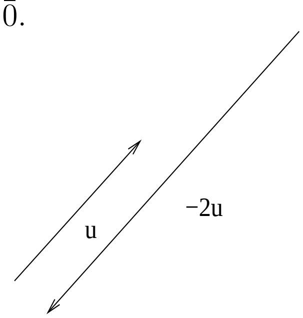
(*i*) Addition

If $\bar{u}$ and $\bar{v}$ are vectors so the initial point of $\bar{v}$ is the end point of $\bar{u}$, we define the sum $\overline{u+v}$ to be the vector with the initial point the same as $\bar{u}$ and the end point the same as $\bar{v}$. In sketches, we have the following:



(*ii*) Scalar Multiplication

If c is a scalar and $\bar{v}$ is a vector, we define the scalar multiple $c\bar{v}$ to be the vector whose magnitude is $|c|||\bar{v}||$ and direction is the same as $\bar{v}$ if $c > 0$, the opposite if $c < 0$. If $c = 0$ or $\bar{v} = 0$, we define $c\bar{v} = \bar{0}$.

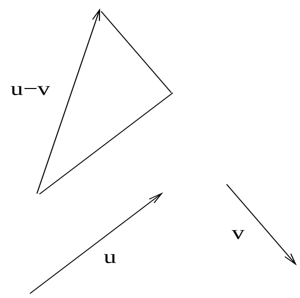


This operation motivates the following definition:

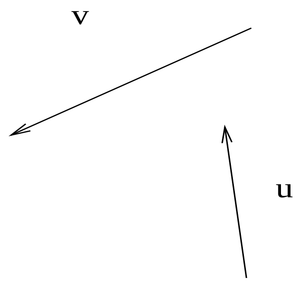
Definition 2.1. We say that two vectors $\vec{v}$ and $\vec{u}$ are parallel if one is a non-zero scalar multiple of the other i.e. there exists a scalar $c \neq 0$ such that $c\vec{u} = \vec{v}$.

(*iii*) Subtraction

For two vectors $\vec{v}$ and $\vec{u}$, we define $\vec{u} - \vec{v}$ as $\vec{u} + (-\vec{v})$, so we "move" down $\vec{u}$ and then "down" $\vec{v}$, but in the opposite direction.



Example 2.2. For the following two vectors $\vec{v}$ and $\vec{u}$, sketch the vectors $2\vec{u} - \vec{v}$ and $\vec{v} + \vec{u}$.



3. Components of Vectors

Obviously drawing a lot of arrows to represent vectors and perform operations is not a practical way of manipulating vectors. We consider a few different ways to represent vectors without having to draw them all the time. We need the following definitions/terminology.

- (i) A vector $\vec{v}$ in 3-space (or 2-space) with its tail at the origin is called a **position vector**.
- (ii) If the head of a position vector $\vec{v}$ points at (a, b, c), we represent it by $\langle a, b, c\rangle$.
- (iii) In general, if $\vec{v}$ is any vector, we denote it by $\langle a, b, c\rangle$ if when its tail is moved to the origin, its head is at the point (a, b, c).
- (iv) We call the entries of the vector $\langle a, b, c\rangle$ the components of $\vec{u}$ - a tells us how far to go in the x-direction, b in the y-direction, and c in the z-direction.
- (v) If we draw a vector $\vec{u} = \langle a, b, c\rangle$ between two points, we call it a geometric representation of the vector $\vec{u}$.
- (vi) Observe that if $\vec{v}$ is represented by an arrow from $P(a, b, c)$ to $Q(d, e, f)$, then the corresponding vector is $\langle d-a, e-b, f-c\rangle$.

Example 3.1. (i) Find the vector represented by the line segment connecting $A(1, 2, 2)$ and $B(3, 3, 2)$.

We have $\vec{AB} = \langle 3-1, 3-2, 2-1\rangle = \langle 2, 1, 1\rangle$.

(ii) How many geometric representations of a vector are there?

Infinitely many (we could have any point as the initial point of the vector).

The convenient thing about this representation is that it makes operations much easier. We summarize:

Result 3.2. Suppose $\vec{u} = \langle a, b\rangle$ and $\vec{v} = \langle a, b, c\rangle$ (2-dimensional vector and a 3-dimensional vector). Then the magnitudes of $\vec{u}$ and $\vec{v}$ are

$$||\vec{u}|| = \sqrt{(a^2 + b^2)}, \text{ and } ||\vec{u}|| = \sqrt{(a^2 + b^2 + c^2)}$$

Result 3.3. The sum, scalar product by C and subtraction of $\vec{u} = \langle a, b, c\rangle$ and $\vec{v} = \langle d, e, f\rangle$ in terms of components are:

$$\vec{u} + \vec{v} = \langle a + d, b + e, c + f\rangle$$

$$C\vec{u} = \langle Ca, Cb, Cc\rangle$$

$$\vec{u} - \vec{v} = \langle a - d, b - e, c - f\rangle$$

A similar result holds for 2-dimensional vectors.

Example 3.4. For $\vec{u} = \langle 2, 3, -5\rangle$ and $\vec{v} = \langle -1, 2, 6\rangle$, find the following:

(i) $2\vec{u} - 3\vec{v}$

We have

$$2\vec{u} - 3\vec{v} = 2 \cdot \langle 2, 3, -5\rangle - 3 \cdot \langle -1, 2, 6\rangle = \langle 4, 6, -10\rangle - \cdot\langle -3, 6, 18\rangle$$
$$= \langle 4 + 3, 6 - 6, -10 - 18\rangle = \langle 7, 0, -28\rangle$$

(ii) $||6\vec{v}||$

We have

$$||6\vec{v}|| = ||6\cdot\langle -1, 2, 6\rangle|| = ||\langle -6, 12, 36\rangle|| = \sqrt{(6^2 + 12^2 + 36^2)} = \sqrt{1476}.$$

The operations of addition and scalar multiplication satisfy many of the standard rules of arithmetic we would expect them to (see the Table on page 810). The only properties you may not be too familiar with are the distributive properties for scalar multiplication over sums of vectors (5,6, and 7).

4. Another Representation

There are many other ways to represent vectors. One particular way which is very popular in physics and the sciences is through the use of unit vectors defined as follows (this definition will be **very** important later on).

Definition 4.1. A unit vector is a vector with length 1.

Result 4.2. If $\vec{u}$ is a nonzero vector, then the unit vector pointing in the same direction as $\vec{u}$ is

$$\frac{1}{||\vec{u}||}\vec{u}.$$

In 3-space (and 2-space), there are three very important unit vectors defined as follows:

Definition 4.3. We define the **standard basis vectors** for $\mathbb{R}^3$ as follows: $\vec{i}$ is the unit vector pointing in the direction of the positive x-axis, $\vec{j}$ is the unit vector pointing in the direction of the positive y-axis, and $\vec{k}$ is the unit vector pointing in the direction of the positive z-axis.

In addition to vector calculus, these vectors will be important in other math classes like linear algebra. Note that $\vec{v} = \langle 1, 0, 0\rangle$, $\vec{i} = \langle 0, 1, 0\rangle$, and $\vec{i} = \langle 0, 0, 1\rangle$, so if $\vec{u} = \langle a, b, c\rangle$, then it can also be written as $a\vec{i} + b\vec{j} + c\vec{k}$. We say that the vector $\vec{v}$ has been written in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$.

Example 4.4. Find a unit vector written in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$ in the direction of $\langle 1, 1, 1\rangle$.

The $\vec{i}$, $\vec{j}$, $\vec{k}$ representation of this vector is $\vec{i} + \vec{j} + \vec{k}$. However, this vector does not have length 1 (in fact, its length is $\sqrt{1+1+1} = \sqrt{3}$). Therefore, to make this is to a unit vector (a process often referred to as **normalizing** the vector, we simply divide by length i.e. a unit vector pointing in the same direction as $\vec{i} + \vec{j} + \vec{k}$ is

$$\frac{\vec{i}}{\sqrt{3}} + \frac{\vec{j}}{\sqrt{3}} + \frac{\vec{k}}{\sqrt{3}}$$

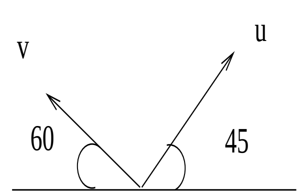
5. Applications

Vectors are extremely useful in many sciences. We look at a couple of examples to illustrate.

Example 5.1. What heading and airspeed are required for a plane to fly due north at 872mph if a wind of 31.5mph is blowing due East.

The resultant vector we want is $\vec{r} = 872\vec{j}$ and the wind vector is $31.5\vec{i}$. Thus the actual speed and direction we need to go is $\vec{s} = 872\vec{j} - 31.5\vec{i}$.

Example 5.2. Two forces, with tension 10 pounds acting on an object as illustrated below. What tension (force) is required to counterbalance these two forces?



We note that $\vec{u} = 10\cos(45)\vec{i} + 10\sin(45)\vec{j}$ and $\vec{v} = -10\cos(60)\vec{i} + 10\sin(60)\vec{j}$, so the force required to counterbalance will be

$$-10(\cos(45) - \cos(60))\vec{i} - 10(\sin(45) + \sin(60))\vec{j}$$

6. What about Multiplication?

We have seen all the properties of addition and scalar multiplication of vectors, but what about a way to "multiply" vectors. We shall consider this idea next!