Section 13.3
The Dot Product

"Multiplying Vectors 1"

We have shown that vectors can be added, and we have shown they can be multiplied by a scalar. We now consider the problem of multiplying two vectors together. There are actually two different ways we shall define to multiply vectors together, each of which have their own individual merits and problems. In this section, we consider the easier of the two called the dot product.

1. Basic Definitions

The dot product is a way to multiply two vectors together to obtain a **scalar** (this is important because multiplication between two objects usually results in a similar object - this is not true with the dot product). There is both a geometric and algebraic definition for the dot product. In order for the geometric definition to make sense, we need to define the angle between two vectors.

Definition 1.1. Suppose $\vec{u}$ and $\vec{v}$ are two non-zero vectos. Then there is a unique plane which contains them and we define the angle between $\vec{u}$ and $\vec{v}$ to be the smallest angle ϑ with $0 \leqslant \vartheta \leqslant \pi$ between $\vec{u}$ and $\vec{v}$ in this plane. If either vector is the zero vector, then we define the angle between them to be $\pi/2$.

Definition 1.2. Suppose $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ and $\vec{v} = d\vec{i} + e\vec{j} + f\vec{k}$ are two vectors in 3-space. Then we define the dot product $\vec{u} \cdot \vec{v}$ as follows:

(*i*) (Geometric Definition) $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos(\vartheta)$ where ϑ is the angle between them.

(*ii*) (Algebraic Definition) $\vec{u} \cdot \vec{v} = ad + be + cf$.

A similar definition for vectors in 2-space holds (we omit the $\vec{k}$).

The dot product is easy to work with.

Example 1.3. Let $\vec{u} = 2\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{v} = \vec{i} - 2\vec{j} - \vec{k}$.

(*i*) Calculate $\vec{u} \cdot \vec{v}$.

We have

$$\vec{u} \cdot \vec{v} = 2 - 6 - 1 = -5.$$

(*ii*) Use this to determine the angle ϑ between $\vec{v}$ and $\vec{u}$.

We know

$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos(\vartheta)$$

so it follows that

$$-5 = \sqrt{(4+9+1)}\sqrt{(1+4+1)}\cos(\vartheta) = \sqrt{14}\sqrt{6}\cos(\vartheta).$$

Therefore, we have

$$\cos(\vartheta) = -\frac{5}{\sqrt{14}\sqrt{6}} \sim -.5455, \text{ or } \vartheta = \arccos(-0.5455) = 2.148.$$

We need two definitions because angles between vectors are usually very difficult to determine, so the geometric definition is difficult to use. However, the algebraic definition is simple to calculate, so we can always use the algebraic definition to determine the dot product and then the geometric definition to determine the angle as we did in the last example. The following result summarizes our idea.

Result 1.4. If ϑ is the angle between nonzero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos(\vartheta) = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}.$$

As we have seen, the dot product is very useful for finding angles between vectors. Recall that two vectors are parallel if they point in the same direction or in opposite directions. This notion can be interpreted in terms of angles. Likewise, we can also define what it means for two vectors to be perpendicular (though we give it a new name). Specifically, we have the following:

Definition 1.5. Two vectors are said to be **parallel** if the angle between them is 0 or π. They are said to be **perpendicular** pr **orthogonal** if the angle between them if $\pi/2$ (we define the zero vector to be orthogonal to all vectors).

There are a number of basic algebraic and geometric properties which the dot product satisfies which are easy to verify:

Result 1.6. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors and c is a scalar, then:

(*i*) $\vec{v}\cdot\vec{v} = ||\vec{v}||^2$
(*ii*) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
(*iii*) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$
(*iv*) $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
(*v*) $\vec{0}\cdot\vec{u} = 0$
(*vi*) $\vec{u}\cdot\vec{v} = 0$ if and only if $\vec{u}$ and $\vec{v}$ are orthogonal.

We illustrate the dot product with some examples.

Example 1.7. Verify the following properties:

(*i*) Property 1

The angle between a vector and itself is 0, so using the geometric definition, we get $\vec{u}\cdot\vec{u} = ||\vec{u}||||\vec{u}||\cos^2(0) = ||\vec{u}||^2$

(*ii*) Property 6

Suppose $\vec{u}\cdot\vec{v} = 0$, we need to show that $\vec{u}$ and $\vec{v}$ are orthogonal. If $\vec{u} = \vec{0}$, then it is orthogonal to all other vectors and in particular to $\vec{v}$. Likewise if $\vec{v} = \vec{0}$. Else, both are nonzero, in which case $||\vec{u}||||\vec{v}|| \neq 0$, so we must have $\cos(\vartheta) = 0$ and hence the angle between them is $\pi/2$, so they are orthogonal.

If they are orthogonal, the angle between them is $\pi/2$, so using the geometric definition of the dot product, we get $\vec{u}\cdot\vec{v} = 0$.

Example 1.8. Determine a numerical value of n which makes the vectors $\vec{u} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$ and $\vec{v} = n\vec{\imath} - 4\vec{\jmath} + 2\vec{k}$ perpendicular.

$\vec{u}$ and $\vec{v}$ are perpendicular if $\vec{u} \cdot \vec{v} = 0$. Using the algebraic definition, we have

$$\vec{u} \cdot \vec{v} = 2n - 12 + 2 = 0 \text{ or } 2n = 10 \text{ so } n = 5.$$

Example 1.9. Find a unit vector perpendicular to $\vec{v} = 2\vec{\imath} + 3\vec{\jmath}$.

Let $\vec{u} = a\vec{\imath} + b\vec{\jmath}$ denote an arbitrary vector. The vector $\vec{u}$ will be perpendicular to $\vec{v}$ if the dot product is 0. Calculating, we have

$$\vec{u} \cdot \vec{v} = 2a + 3b = 0, \text{ or } a = -\frac{3b}{2}.$$

It follows that for any value of b, the vector

$$\vec{u} = -\frac{3b}{2}\vec{\imath} + b\vec{\jmath}$$

will be perpendicular to $\vec{v}$. To make $\vec{v}$ a unit vector, we need to make $||\vec{v}|| = 1$. Calculating, we have

$$||\vec{v}|| = \sqrt{\left(\frac{9b^2}{4} + b^2\right)} = 1, \text{ or } b = \frac{\sqrt{13}}{2} = 1, \text{ so } b = \frac{2}{\sqrt{13}}.$$

Therefore, a unit vector perpendicular to $\vec{v}$ is

$$\vec{u} = -\frac{3}{\sqrt{13}}\vec{\imath} + \frac{2}{\sqrt{13}}\vec{\jmath}.$$

Example 1.10. If $\vec{v} = 3\vec{\imath} + 2\vec{\jmath} - \vec{k}$, find a vector $\vec{u}$ such that $\vec{u} \cdot \vec{v} = 1$.

Let $\vec{u} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ denote an arbitrary vector. Then $\vec{u}\cdot\vec{v} = 3a + 2b - c$. This is equal to 1 for any a, b, c satisfying $3a + 2b - c = 1$, so we can choose $a = 0$, $b = 0$, and $c = -1$ i.e. $\vec{u} = -\vec{k}$.

2. Direction Angles

The **direction angles** of a vector $\vec{u}$ are the angles $\vec{u}$ makes with the three positive coordinate axis respectively - α the angle with the positive x-axis, β the angle with the positive y-axis, and γ the angle with the positive z-axis. Since $\vec{\imath}$ points in the direction of the positive x axis, α is simply the angle $\vec{u}$ makes with $\vec{\imath}$, and likewise β is simply the angle $\vec{u}$ makes with $\vec{\jmath}$ and γ is simply the angle $\vec{u}$ makes with $\vec{k}$, and these are easy to calculate using our results. In a similar theme, we define the direction cosines to be the cosines of the angles of $\vec{u}$ made with each coordinate axis. Again, calculations of these values are straight forward using our previous observations. We summarize.

Result 2.1. The direction angles a vector $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ are the angles $\vec{u}$ makes with each coordinate axis. The direction cosines are the cosines of the angles $\vec{u}$ makes with the coordinate axis. If α, β and γ denotes the direction angles in the x, y and z-directions respectively, the cosines and angles can be calculated as:

$$\cos\alpha = \frac{a}{||\vec{u}||}, \cos\beta = \frac{b}{||\vec{u}||}, \cos\alpha = \frac{c}{||\vec{u}||}$$

Example 2.2. (*i*) Show the formula for the direction cosine in the direction of $\vec{i}$ holds.

Let $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$. To calculate the direction cosine with $\vec{i}$, we can use the geometric and algebraic definition of the dot product. Specifically, we have

$$||\vec{u}||||\vec{i}||\cos(\alpha) = \vec{u}\cdot\vec{i} = (a\vec{i} + b\vec{j} + c\vec{k})\cdot\vec{i} = a.$$

Therefore,

$$\cos(\vec{u}) = \frac{a}{||\vec{u}||}.$$

(*ii*) Find the direction angles for $\vec{v} = 2\vec{i} - \vec{j} + \vec{k}$.

First, we note that $||\vec{v}|| = 6$, so $\cos(\alpha) = \frac{2}{6}$, $\cos(\beta) = -\frac{1}{6}$ and $\cos(\gamma) = \frac{1}{6}$.

(*iii*) Show that $\cos(\alpha)^2 + \cos(\beta)^2 + \cos(\gamma)^2 = 1$.

We have

$$\cos(\alpha)^2 + \cos(\beta)^2 + \cos(\gamma)^2 = \frac{a^2}{||\vec{u}||^2} + \frac{b^2}{||\vec{u}||^2} + \frac{c^2}{||\vec{u}||^2} = \frac{a^2+b^2+c^2}{a^2+b^2+c^2} = 1.$$

3. PROJECTIONS

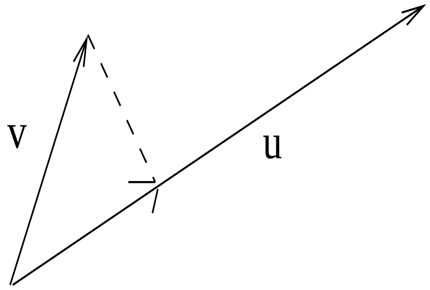
When we write a vector $\vec{u}$ in component form: $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$, the scalars a, b and c tell us how far we must move in each direction i.e. we move a distance a in the $\vec{i}$ direction, and all other movement is in a perpendicular direction to $\vec{i}$. We usually call the scalars a, b and c the projection of $\vec{u}$ onto $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively. This idea can be generalized to vectors which are not coordinate vectors. Specifically, we define the following:

Definition 3.1. Suppose $\vec{u}$ and $\vec{v}$ are vectors. We define the **vector projection** of $\vec{v}$ onto $\vec{u}$ to be the vector $\vec{v}_{\text{par}}$ such that $\vec{v}_{\text{par}}$ is parallel to $\vec{u}$, and there exists a vector $\vec{v}_{\text{parp}}$ which is perpendicular to $\vec{u}$ and

$$\vec{v} = \vec{v}_{\text{per}} + \vec{v}_{\text{parp}}.$$

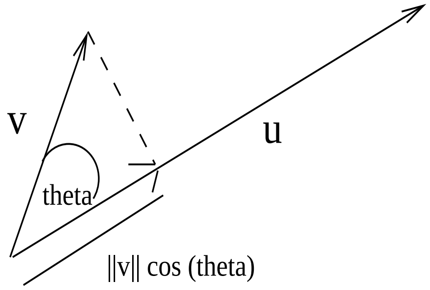
In words, the vector projection of $\vec{v}$ onto $\vec{u}$ is the component of $\vec{v}$ pointing in the same direction as $\vec{u}$. We define the **scalar projection** of $\vec{v}$ onto $\vec{u}$ to be the magnitude of the vector projection: $||\vec{v}_{\text{par}}||$.

For two dimensional vectors, we illustrate the vector projection of $\vec{v}$ onto $\vec{u}$ below:



In order to project one vector $\vec{v}$ onto another $\vec{u}$, we do the following:

- Let $\vec{v}_{\text{par}}$ and $\vec{v}_{\text{perp}}$ denote the components of $\vec{v}$ parallel and perpendicular to $\vec{u}$ respectively, so $\vec{v} = \vec{v}_{\text{par}} + \vec{v}_{\text{perp}}$.
- Normalize $\vec{u}$ - i.e. make it into a unit vector and rename it $\vec{u}_{\text{unit}}$.
- If the angle ϑ between $\vec{v}$ and $\vec{u}$ is between 0 and $\pi/2$, we have $||\vec{v}_{\text{par}}|| = ||\vec{v}\cos(\vartheta)||$ (see picture)



- Since $\vec{u}_{\text{unit}}$ is a unit vector, this can be calculated as
$$||\vec{v}_{\text{par}}|| = \vec{v} \cdot \vec{u}_{\text{unit}}$$
i.e
$$\vec{v} \cdot \vec{u}_{\text{unit}} = ||\vec{v}||||\vec{u}_{\text{unit}}|| \cos(\vartheta) = ||\vec{v}|| \cos(\vartheta)$$
- Since it points in the same direction as $\vec{u}_{\text{unit}}$, we get
$$\vec{v}_{par} = (\vec{v} \cdot \vec{u}_{\text{unit}})\vec{u}_{\text{unit}}$$
(a similar argument works when $\vartheta > \pi/2$).
- To find $\vec{v}_{\text{perp}}$, we calculate $\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{par}}$.

Summarizing, we have the following:

Result 3.2. If $\vec{u}$ and $\vec{v}$ are vectors, then:

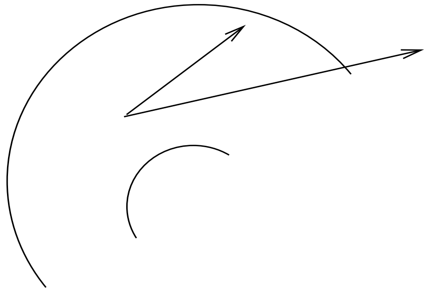
(i) The scalar projection (the magnitude of the projection) of $\vec{v}$ onto $\vec{u}$ is
$$||\vec{v}_{\text{par}}|| = comp_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{||\vec{u}||}$$

(ii) The vector projection of $\vec{v}$ onto $\vec{u}$ is

$$\vec{v}_{\text{par}} = proj_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\vec{u}$$

Projections are important in many different applications in fields such as physics. We illustrate.

Example 3.3. A car is turning a corner. Its velocity vector is $\vec{v} = 3\vec{i} + 2\vec{j}$. If it is traveling too fast, it will drive off the road. Specifically, if its magnitude in the direction of $\vec{u} = \vec{i} + \vec{j}$ is greater than 5, it will skid off the road. Will the car drive of the road?



We just need to check the scalar projection of $\vec{v}$ onto $\vec{i} + \vec{j}$. We have

$$||\vec{v}_{\text{par}}|| = \frac{\vec{v}\cdot\vec{u}}{||\vec{u}||} = \frac{5}{\sqrt{2}} < 5$$

so the car will be safe.

Another application of vectors and projections is work and force. Work done is calculated by multiplying the force by the distance. If a force is acting on an object by pushing or pulling, in the direction an object is moving, then the work is calculated as $F \times d$ (force times distance). However, it is conceivable that the direction of a force may be different to the direction in which can object moves (for example, pushing an cart in a slight downward direction will still result in horizontal movement). In this case, the work done on moving the object will not simply be the force times the distance since the whole force is not acting on the object, only the component of the force in the same direction that the object is moving. It follows that if a force has vector $\vec{F}$ and moves an object along the vector $\vec{d}$, the work done will be

$$\vec{F}_{\text{par}}\cdot\vec{d} = \frac{\vec{F}\cdot\vec{d}}{||\vec{d}||^2}\vec{d}\cdot\vec{d} = \vec{F}\cdot\vec{d}.$$

Summarizing, we have:

Result 3.4. The work done by a constant force $\vec{F}$ moving an object along the vector $\vec{d}$ is $\vec{F}\cdot\vec{d}$.

We finish with an example to illustrate.

Example 3.5. A constant force $\vec{F} = 10\vec{i} + 18\vec{j} - 6\vec{k}$ moves an object from $(2, 3, 0)$ to $(4, 9, 15)$. What is the work done by this force?

We note that $\vec{d} = 2\vec{i} + 6\vec{j} + 15\vec{k}$, so work done is $\vec{F} \cdot \vec{d} = 20 + 108 - 90 = 38J$ (work is measured in Joules).