Section 13.4
The Cross Product

"Multiplying Vectors 2"

In this section we consider the more technical multiplication which can be defined on vectors in 3-space (but not vectors in 2-space).

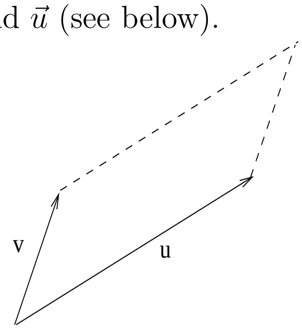
1. Basic Definitions

The cross-product of two vectors in 3-space is a way to "multiply" two vectors and obtain a third vector - this is one major difference compared to the dot product where the result is a scalar. As with the dot product, there are different ways to define the cross-product. The first definition we consider is more of a description, and the next two give us ways of calculating the cross-product via geometry and algebra.

Definition 1.1. Given two vectors $\vec{u}$ and $\vec{v}$, we define the cross-product $\vec{u} \times \vec{v}$ to be $\vec{0}$ if they are parallel or the vector satisfying the following two properties if they are not.

(i) the direction of $\vec{u} \times \vec{v}$ is in the direction of the vector $\vec{n}$ orthogonal to both $\vec{u}$ and $\vec{v}$ determined by the right hand rule i.e. we point our index finger in the direction of $\vec{u}$, our middle finger points in the direction of $\vec{v}$, though closer to the palm of our hand than the index finger, and our thumb points perpendicular to these two fingers - then $\vec{n}$ points in the direction of the thumb.

(ii) the magnitude of $\vec{u} \times \vec{v}$ is equal to the area of the parallelogram formed by $\vec{v}$ and $\vec{u}$ (see below).



Of course, though descriptive, this definition does not provide us with a way to actually calculate the cross product. For this, we have the following two definitions:

Definition 1.2. Suppose $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ and $\vec{v} = d\vec{i} + e\vec{j} + f\vec{k}$ are vectors. Then we define $\vec{u} \times \vec{v}$ as follows:

(i) (Geometric Definition)

$$\vec{u} \times \vec{v} = (||\vec{u}||||\vec{v}|| \sin(\vartheta))\vec{n}$$

where $\vec{n}$ is the unit vector pointing in the direction of the vector determined by the right hand rule.

(*ii*) (Algebraic Definition)

$$\vec{u} \times \vec{v} = (bf - ce)\vec{i} + (cd - af)\vec{j} + (ae - bd)\vec{k}$$

The algebraic definition is really the only definition which allows us to calculate the cross-product, but it is a fairly technical definition to remember. In order to remember, instead you can just remember it using determinants of matrices. Specifically, we have

$$\vec{u} \times \vec{v} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix}.$$

Also, the calculator can be used to determine the cross product of any two vectors. We illustrate with a couple of examples.

Example 1.3. Suppose $\vec{u} = 2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{v} = -\vec{j} + 2\vec{k}$ and $\vec{w} = -\vec{i} + \vec{j}$. Determine the following:

(*i*) $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 0 & -1 & 2 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

$$= (6+1)\vec{i} - (4)\vec{j} + (-2)\vec{k} = 7\vec{i} - 4\vec{j} - 2\vec{k}$$

(*ii*) $\vec{v} \times \vec{u}$

$$\vec{v} \times \vec{u} = -7\vec{i} + 4\vec{j} + 2\vec{k}$$

(*iii*) $(\vec{u} \times \vec{v}) \cdot \vec{w}$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (7\vec{i} - 4\vec{j} - 2\vec{k}) \cdot (-\vec{i} + \vec{j}) = -7 - 4 = -11$$

(*iv*) $\vec{u} \times (\vec{v} \cdot \vec{w})$

Doesn't make any sense - you cannot cross a scalar with a vector!

The cross-product satisfies many important algebraic identities as outlined below.

Result 1.4. (Identities) Suppose $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in 3-space and c is a scalar.

(*i*) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ (skew-symmetric)

(*ii*) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

(*iii*) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(*iv*) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

(*v*) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

(vi) $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

Each of these identities can be checked using the formula (though they take a lot of time and patience. Note that the cross-product is neither commutatative or associative!!! With all the different definitions we have given for the cross-product, there are a number of important properties we can derive. We summarize them.

Result 1.5. (Properties) Suppose $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors.

(i) The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
(ii) $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin(\vartheta)$
(iii) Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$
(iv) The area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is $||\vec{u}||||\vec{v}|| \sin(\vartheta)$.
(v) The area of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$ is $||\vec{u} \cdot (\vec{v} \times \vec{w}||$.

We examine some of these results and their applications.

Example 1.6. (i) Find a vector orthogonal to the plane through the points $P(1, 0, 0)$, $Q(1, 2, 3)$ and $R(2, 7, -9)$ (this will be an important tool in the following sections)

To find an orthogonal vector, we use the cross product. First, we need to determine two vectors in the plane. We can use the displacement vectors between the points - $\vec{u} = 2\vec{j} + 3\vec{k}$ and $\vec{v} = \vec{i} + 7\vec{j} - 9\vec{k}$. Then $\vec{u} \times \vec{v} = -39\vec{i} + 3\vec{j} - 2\vec{k}$ is a vector orthogonal to both of these (by the definition of the cross-product) and hence the plane.

(ii) Suppose $\vec{a} \neq \vec{0}$. Is it true that if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ then $\vec{b} = \vec{c}$?

No - take $\vec{a} = \vec{i}$, $\vec{b} = \vec{i}$ and $\vec{c} = 2\vec{i}$.

(iii) Suppose $\vec{a} \neq \vec{0}$. Is it true that if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ then $\vec{b} = \vec{c}$?

No - take $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{i}$ and $\vec{c} = \vec{j}$.

(iv) CHALLENGE (POSSIBLE EXTRA CREDIT POINTS) - Show that for nonzero $\vec{a}$, if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ then $\vec{b} = \vec{c}$?

(v) Find the area of the triangle with vertices $P(1, 2, 3)$, $Q(2, 8, 9)$ and $R(-1, -2, 3)$.

First we find the vectors representing the sides: $\vec{u} = \vec{i} + 6\vec{j} + 6\vec{k}$ and $\vec{v} = 2\vec{i} + 4\vec{j}$. The area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ is equal to $||\vec{u} \times \vec{v}||$ so the area of the triangle will be

$$||\vec{u} \times \vec{v}||/2 = || - 24\vec{i} + 12\vec{j} - 8\vec{k}||/2 = 14.$$

(vi) Find a non-zero vector $\vec{s}$ such that $\vec{u} \times \vec{s} = \vec{0}$.

We can attempt to do this directly. Let $\vec{s} = a\vec{i} + b\vec{j} + c\vec{k}$ be an arbitrary vector. Then we have

$$\vec{u} \times \vec{s} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ a & b & c \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 1 \\ b & c \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ a & c \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 3 \\ a & b \end{vmatrix}$$

$$= (3c - b)\vec{i} - (2c - a)\vec{j} + (2b - 3a)\vec{k}.$$

For this to equal $\vec{0}$, we need $3c - b = 0$, $2c - a = 0$ and $2b - 3a = 0$, so we have a linear system if 3 equations to solve. This is a fairly difficult problem which suggests this is not the to solve this problem.

Notice however that by Property (iii), that two cross product of two non-zero vectors is the zero vector if and only if they are parallel. In particular, we do not need to solve this system of equations - we simply need to exhibit a vector which is parallel to $\vec{u}$, so we can choose for example $\vec{s} = 4\vec{i} + 6\vec{j} + 2\vec{k} = 2\vec{u}$.

2. The Cross-Product on the Standard Basis Vectors

Another way to remember the cross product is to use the distributive properties over addition of vectors and how the cross product affects the basis vectors. Specifically, we have

$$\begin{array}{lll} \vec{i} \times \vec{i} = \vec{0} & \vec{i} \times \vec{j} = \vec{k} & \vec{j} \times \vec{i} = -\vec{k} \\ \vec{i} \times \vec{k} = -\vec{j} & \vec{k} \times \vec{i} = \vec{j} & \vec{k} \times \vec{k} = \vec{0} \\ \vec{j} \times \vec{k} = \vec{i} & \vec{k} \times \vec{j} = -\vec{i} & \vec{j} \times \vec{j} = \vec{0} \end{array}$$

We illustrate with an example.

Example 2.1. Use the distributive properties of the cross product and the cross product on the basis vectors to determine $(3\vec{i} + 2\vec{j}) \times (2\vec{j} + \vec{k})$.

We have

$$(3\vec{i} + 2\vec{j}) \times (2\vec{j} + \vec{k}) = 3\vec{i} \times (2\vec{j} + \vec{k}) + 2\vec{j} \times (2\vec{j} + \vec{k})$$

$$= 6\vec{i} \times \vec{j} + 3\vec{i} \times \vec{k} + 4\vec{j} \times \vec{j} + 2\vec{j} \times \vec{k} = 6\vec{k} - 3\vec{j} + 2\vec{i}$$

3. Torque

If we apply a force $\vec{F}$ on a rigid body at a point given by a position vector $\vec{r}$ (such as turning a wrench), the torque τ is defined to be the cross product $\vec{r} \times \vec{F}$ and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation and the magnitude is the force rotating it. We illustrate a typical example.

Example 3.1. A wrench 30cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $3\vec{j} - 4\vec{k}$ to the end of the wrench. Find the magnitude of the force needed to supply 100 joules of torque to the bolt.

The force is applied at the end of the wrench, so will have position vector $0.3\vec{j}$. We know the force at this point is in the direction of $3\vec{j} - 4\vec{k}$, so the force vector $\vec{F}$ will be of the form $3A\vec{j} - 4A\vec{k}$ for some constant A. Calculating torque, we have $\tau = 1.2A\vec{i}$, and this has magnitude $1.2A$. In order for this to equal 100, we must have $A = 100/1.2 = 250/3$. Thus we have

$$||\vec{F}|| = ||250\vec{j} - 1000\vec{k}/3|| = \sqrt{(250^2 + (1000/3)^2)} \cong 417$$