# Section 13.6
# Equations of Cylinders and Quadric Surfaces

"Other Graphs in 3-space"

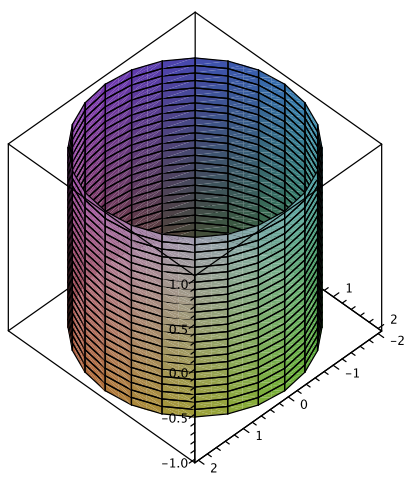
In this section we shall consider other types of surfaces. In order to sketch such curves, we consider **cross-sections** with planes (also called **traces**).

## 1. Cylinders

Though we have previously considered cylinders as a surface shaped like a pipe, in multivariable calculus, there is a more general definition of a cylinder, and there are many other graphs which are considered "cylinders" according to the formula definition which do not look like pipes. The formal definition of a cylindrical surface is the following:

**Definition 1.1.** A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given curve in some plane.

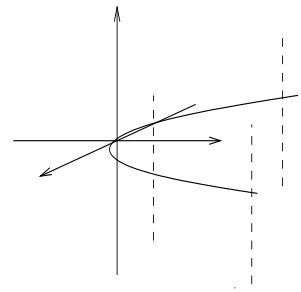
To avoid confusion, we shall usually refer to the cylinders which are shaped like a pipe as **pipe cylinders**, and general cylinders satisfying this definition a cylindrical surface (or cylinder for short). Of course, a pipe cylinder is a cylindrical surface i.e. it consists of parallel lines, all of which all pass through a circle (see illustration below).




There are many other examples of cylindrical surfaces.

**Example 1.2.** Sketch the graph of $y = x^2$ (in 3-space). Is it a cylinder? We have to be careful not to draw interpret it as an equation in 2-space even though there is no $z$ - that just means it does not depend upon $z$. If we sketch this graph in the $xy$-plane (so $z = 0$, we obtain the parabola $y = x^2$. Since there is no $z$-value in the equation, we shall get the same curve in every possible plane parallel to the $xy$-plane. In

particular, the graph of this surface will be all vertical lines passing through the curve $y = x^2$ in the $xy$-plane. By definition, this makes the graph a cylinder.



**Remark 1.3.** As a general case, if one variable is missing from an equation, then the corresponding graph will be a cylindrical surface.

## 2. Quadric Surfaces

We have seen that linear equations in 3-space have graphs which are planes. The next easiest type of equation to study in single variable is the quadratic, or second degree. Similarly, in 3-space, the second easiest equation to study is a second degree.

**Definition 2.1.** A quadric surface is the graph of a second degree equation in 3 variables

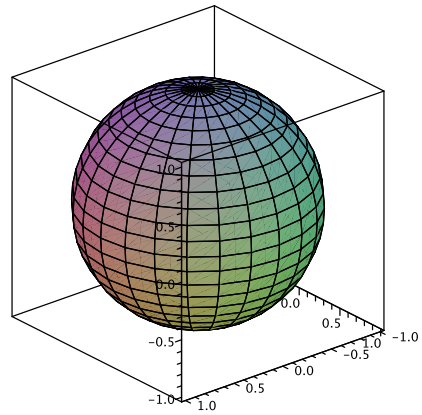
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + iZ + J = 0$$

for constants $A$ through $J$.

It is a fact that by translating the graph (meaning rigidly moving it), it can be rewritten in the form $Ax^2+by^2+Cz^2+J = 0$ or $Ax^2+By^2+Iz = 0$ for constants $A$, $B$, $C$, $I$, and $J$. Therefore, to understand quadric surfaces, we only need to study these types of surfaces. In total, there are six types of graphs for these surfaces which can be found on page 844 of your text book - you should be able to recognize each from the equations. We shall finish by illustrating how to classify a quadric surface using these graphs (note that it will require completing the square), and by using cross-sections.

**Example 2.2.** Sketch the graph of the quadric surface $x^2 + y^2 + z^2 = 1$. What happens if you change the constants (though they remain positive) in front of the $x^2$, $y^2$ and $z^2$?

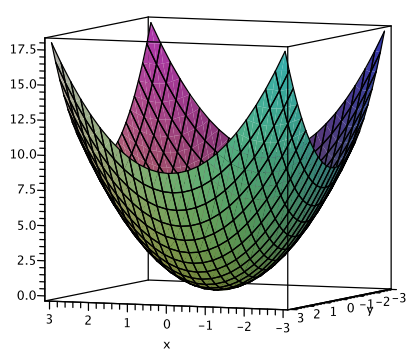
This is just the graph of a sphere of radius 1 centered at the origin.




If you change the constants, it stretches or compresses the sphere along the relevant axis making it into an ellipsoid. For example, if we multiply the $x^2$ by 4, then the $x$-values will only lie between $-1/2$ and $1/2$ and thus the graph would be an ellipsoid with a smaller radii in the $x$-direction.

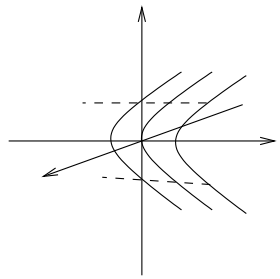
**Example 2.3.** Describe and sketch the quadric surface $z = x^2 + y^2$.

For any fixed value of $z = k \geqslant 0$, in the plane $z = k$, the trace (or cross-section) is a circle of radius $k$. There are no solutions for $z < 0$. This means that as $z$ increases, the circles which make up the surface get larger, but at a slower rate. The result is something called a elliptic paraboloid as illustrated below.




**Example 2.4.** Draw the trace lines of the quadric surface $4y = x^2 + z^2$ with $x = k$ a constant.

If we fix $x$, we get an equation of the from $4y = k^2 + z^2$ which is a parabola (of $y$ in terms of $z$). As $x$ increases, this parabola moves up and as $z$ decreases, this parabola moves down. It suggests a graph of the following type:



**Example 2.5.** Reduce the equation $z^2 = 4x^2 + 9y^2 + 36$ to one of the standard forms so you can classify it (meaning determine what its graph looks like).

This looks like a hyperboloid of 2 sheets. We check:

$$z^2 = 4x^2 + 9y^2 + 36 \to -4x^2 - 9y^2 + z^2 = 36 \to -\frac{4x^2}{36} - \frac{9y^2}{36} + \frac{z^2}{36} = 1$$
$$\to -\frac{x^2}{3^2} - \frac{y^2}{2^2} + \frac{z^2}{6^2} = 1$$

which is precisely the equation for a hyperboloid of two sheets.

**Example 2.6.** Classify the graph of the equation $z^2 = x^2+2x-9y^2+37$ as a translation of one of standard forms so you can classify it (meaning determine what its graph looks like).

Completing the square in $x$, we get

$$z^2 = (x+1)^2 - 9y^2 + 36.$$

Using simple algebra, we get

$$z^2 = (x+1)^2-9y^2+36 \to -(x+1)^2+9y^2+z^2 = 36 \to -\frac{(x+1)^2}{36}+\frac{9y^2}{36}+\frac{z^2}{36} = 1$$
$$\to -\frac{(x+1)^2}{6^2} + \frac{y^2}{2^2} + \frac{z^2}{6^2} = 1.$$

This is a hyperboloid of two sheets with center shifted to $(-1, 0, 0)$ and opening out in the $x$-direction instead of the $z$-direction (since the roles of $x$ and $z$ have been switched.