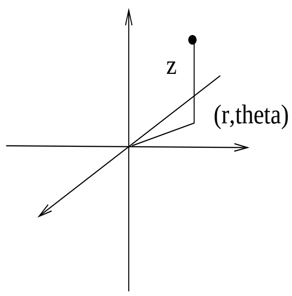
Section 13.7
Cylindrical and Spherical Coordinates

"Non-Rectangular Coordinate Systems in 3-space"
In Calculus II, we considered the polar coordinate system to help integrate functions whose graphs were circular regions. In this section, we consider two new coordinate systems for graphs in 3-space.

1. Cylindrical Coordinates

The first coordinate system we consider is a generalization of polar coordinates - the basic idea is to take the polar coordinates in the xy-plane and then simply add the z-coordinate to determine the height of a point. They are particularly useful when describing cylinders. Formally, we define the cylindrical coordinate system as follows.

Definition 1.1. The cylindrical coordinates of a point P in 3-space is defined to be (r, ϑ, z) where (r, ϑ) are the polar coordinates of the projection of P in the xy-plane and z is the z-coordinate of the plane where $r \geqslant 0$ and $0 \leqslant \vartheta < 2\pi$.

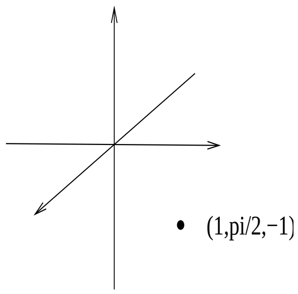


Since cylindrical coordinates are so closely related to polar coordinates, it is easy to convert from rectangular coordinates in 3-space into cylindrical and vice versa.

Result 1.2. (*i*) The rectangular coordinates of the point (r, ϑ, z) in 3 space are $x = r\cos(\vartheta)$, $y = r\sin(\vartheta)$ and $z = z$.
(*ii*) The cylindrical coordinates of the point (x, y, z) can be found by solving the equations $x^2+y^2 = r^2$, $\tan(\vartheta) = y/x$ and $z = z$.

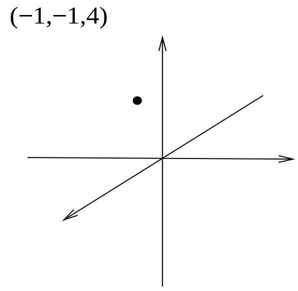
Remember that when converting from rectangular to polar, you need to be very careful with the angle ϑ because simply applying the inverse of tan on a calculator will not necessarily give you the correct answer (WHY?). We illustrate with a few examples.

Example 1.3. Plot the point with cylindrical coordinates $(1, \pi/2, -1)$ and convert to rectangular coordinates.



Converting, we have $x = 0$, $y = 1$ and $z = -1$, so $(0, 1, -1)$.

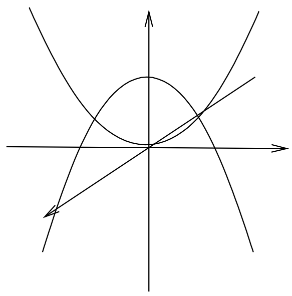
Example 1.4. Plot the point with rectangular coordinates $(-1, -2, 4)$ and convert to cylindrical coordinates.



Converting, we have $r^2 = 1+1 = 2$, so $r = \sqrt{2}$, $\tan(\vartheta) = 1$, so $\vartheta = \pi/4$ or $3\pi/4$ and $z = 4$. To determine ϑ, we just observe in the diagram it cannot be $\pi/4$, so the coordinates are $(1, 3\pi/4, 4)$.

Example 1.5. Describe the surface $r^2 + z^2 = 1$.
Substituting $r^2 = x^2 + y^2$, we see that this is simply the equation for a sphere of radius 1 centered at the origin.

Example 1.6. Sketch the region bounded by $r^2 \leqslant z \leqslant 2 - r^2$
To do this, we first set the equations equal giving the equations $r^2 = z$ and $z = 2 - r^2$. Both of these are equations for elliptic paraboloids, one with base at the origin centered in the z action pointing in the negative z-direction, and the other has its base in the z-axis at $z = 2$ and is pointing upward. We illustrate a the two-dimensional cut (or trace) with the yz-plane.



Example 1.7. Write the equation $y = 5$ in polar coordinates.

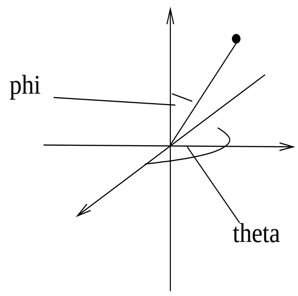
We know $y = r\sin(\vartheta)$, so we must have $r\sin(\vartheta) = 5$ as the polar equation for this rectangular function (there are no restrictions on z or x).

2. Spherical Coordinates

The second set of coordinates we consider are a little more complicated. They are particularly useful when describing regions or surfaces which are similar to a sphere.

Definition 2.1. We define the spherical coordinates $(\varrho, \vartheta, \varphi)$ of a point P in space as follows:

- ϱ is the distance of P from the origin (so $\varrho \geqslant 0$).
- ϑ is the same angle as in cylindrical coordinates (the angle made from the positive x-axis, so $0 \leqslant \vartheta < 2\pi$).
- φ is the angle between the positive z-axis and the line segment connecting P to the origin (so $0 \leqslant \varphi \leqslant \pi$).



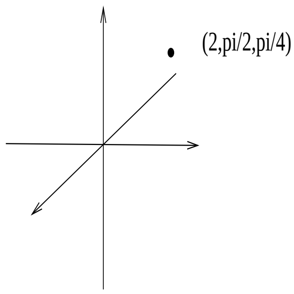
Conversion between these coordinate systems is a little more complicated. It is done using the following formulas.

Result 2.2. (*i*) The rectangular coordinates of the point $(\varrho, \vartheta, \varphi)$ in 3 space are $x = \varrho\sin(\varphi)\cos(\vartheta)$, $y = \varrho\sin(\varphi)\sin(\vartheta)$ and $z = \varrho\cos(\varphi)$.

(*ii*) The spherical coordinates of the point (x, y, z) can be found by solving the equation $\varrho^2 = x^2 + y^2 + z^2$, and then using the equations given above.

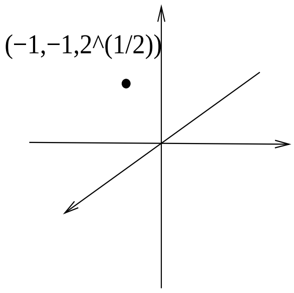
Remember that when converting from rectangular to spherical, you need to be very careful with the angles ϑ and φ (WHY?). We illustrate with a few examples.

Example 2.3. Plot the point with spherical coordinates $(2, \pi/2, \pi/4)$ and convert to rectangular coordinates.



Converting, we have $x = 0$, $y = \sqrt{2}$ and $z = \sqrt{2}$, so $(0, \sqrt{2}, \sqrt{2})$.

Example 2.4. Plot the point with rectangular coordinates $(-1, -1, \sqrt{(2)})$ and convert to spherical coordinates.



Converting, we have $\varrho^2 = 1 + 1 + 2 = 4$, so $\varrho = 2$. Then we have $\sqrt{2} = 2\sin(\varphi)$, so $\sin(\varphi) = \sqrt{2}/2$, or $\varphi = \pi/4$. Finally, $-1 = 2\sin(\varphi)\cos(\vartheta)$ and $-1 = 2\sin(\varphi)\sin(\vartheta)$ giving $\sin(\varphi) = \cos(\varphi) = -\sqrt{2}/2$, or $\vartheta = 5\pi/4$. So the coordinates are $(2, 5\pi/4, \pi/4)$.

Example 2.5. Write the equation $z = x^2 + y^2$ in spherical coordinates. We know $x^2 + y^2 + z^2 = \varrho^2$, so we can replace the right hand side by $\varrho^2 - z^2$ giving $z = \varrho^2 - z^2$ or $z^2 + z = \varrho^2$. Since $z = \varrho\cos(\varphi)$, we have $\varrho^2\cos^2(\varphi) + \varrho\cos(\varphi) = \varrho^2$, so simplifying, we get

$$\varrho\cos^2(\varphi) + \cos(\varphi) = \varrho.$$

Example 2.6. Sketch the solid bounded by $0 \leqslant \varphi \leqslant \pi/4$, and $\varrho \leqslant 1$. We can describe this solid in words - the angle from the z-axis is bounded from 0 to $\pi/4$ and the distance from the origin is less than or equal to 1, and there are no restrictions on ϑ. This suggests a shape like an ice-cream cone as illustrated below: