

Section 14.2

Derivatives and Integrals of Vector Functions

“Calculus for vectors”

In this section, we generalize the ideas from Calc 1 and 2 to vector valued functions.

1. DERIVATIVES

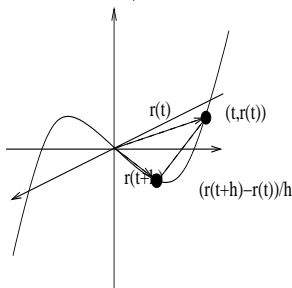
We define the derivative of a vector function in exactly the same way as a scalar valued function - using a difference quotient.

Definition 1.1. We define the derivative of a vector function $\vec{r}(t)$ and denote it as follows:

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

provided the limit exists. If the limit exists everywhere, the curve is said to be smooth.

The value of the derivative for scalar valued functions coincides with the slope of a graph at a point P . There is a similar interpretation for vector valued functions. Recall that a vector valued function defines a curve C in 3-space (by placing the tail of each vector at the origin - position vectors). Under this realization, for any value of t at the point $P = \vec{r}(t)$, the derivative vector $\vec{r}'(t)$ points in the direction of the tangent line of the curve C at P (see illustration below).



For this reason, we call the derivative $\vec{r}'(t)$ the tangent vector of C provided it exists and $\vec{r}'(t) \neq \vec{0}$. We also define the tangent line to be the line which passes through P and points in the direction of $\vec{r}'(t)$. Since vectors can have varying length, there are in fact infinitely many different tangent vectors to a curve at a given point. Therefore, to avoid ambiguity, we often consider tangent vectors with length 1. Specifically, we define the following:

Definition 1.2. We define the unit tangent vector to a vector function $\vec{r}(t)$ to be the vector $\vec{T}(t)$ defined by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

We summarize our remarks.

Remark 1.3. The derivative of a vector function $\vec{r}'(t)$ at $t = a$ is a vector pointing in the direction of the tangent line to the space curve traced out by $\vec{r}(t)$ at $t = a$.

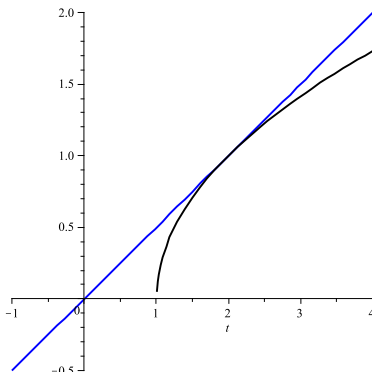
Computing the derivative of a vector valued function at a point is easy in component form (provided you know how to calculate the derivative of a scalar valued functions!).

Result 1.4. If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ and f , g , and h are differentiable, then $\vec{r}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$

To illustrate, we look at some examples.

Example 1.5. Sketch the curve $\vec{r}(t) = (1+t)\vec{i} + \sqrt{t}\vec{j}$, find $\vec{r}'(t)$ and then draw the tangent line at $t = 1$.

We have $x = 1 + t$, or $t = x - 1$, so $y = \sqrt{x - 1}$. Calculating, $\vec{r}'(t) = \vec{i} + t^{-1/2}/2\vec{j}$, so at $t = 1$, we have $\vec{r}'(1) = \vec{i} + \vec{j}/2$. At $t = 1$, the point P on the curve is $(2, 1)$, so sketching, we have:

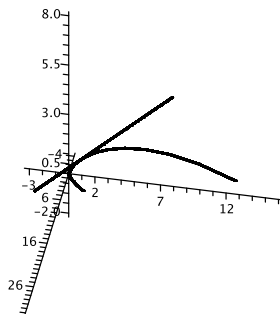


Example 1.6. Find the unit tangent vector to $\vec{r}(t) = \cos(t)\vec{i} + 3t\vec{j} + 2\sin(2t)\vec{k}$ at $t = 0$.

We have $\vec{r}'(t) = \sin(t)\vec{i} + 3\vec{j} - 4\cos(t)\vec{k}$, so $\vec{r}'(0) = 3\vec{j} - 4\vec{k}$. Thus we have $\vec{T}(t) = (3\vec{j} - 4\vec{k})/(\sqrt{3^2 + 4^2}) = 3\vec{j}/5 - 4\vec{k}/5$.

Example 1.7. Find parametric equations for the tangent line to the curve with parametric equations $x(t) = t^5$, $y(t) = t^4$, $z(t) = t^3$ at the point $(1, 1, 1)$ and sketch the graph and the tangent line.

The vector equation for this curve is $\vec{r}(t) = t^5\vec{i} + t^4\vec{j} + t^3\vec{k}$. Its derivative at $P(1, 1, 1)$ is $5\vec{i} + 4\vec{j} + 3\vec{k}$. Since it passes through the point $(1, 1, 1)$, parametric equations for this line will be $x(t) = 1 + 5t$, $y(t) = 1 + 4t$, and $z(t) = 1 + 3t$. Equivalently, the vector equation of the tangent line will be $\vec{l}(t) = (1 + 5t)\vec{i} + (1 + 4t)\vec{j} + (1 + 3t)\vec{k}$. The graph looks like the following:



Example 1.8. The curves $\vec{r}_1(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ and $\vec{r}_2(t) = \sin(t)\vec{i} + \sin(2t)\vec{j} + t\vec{k}$ intersect at the origin. Find their angle of intersection.

We just need to find the angle between the two vectors which point in the same direction as the tangent lines. We first observe that these two lines pass through the origin when $t = 0$, so we differentiate each and evaluate at $t = 0$ to get the vector of the tangent line of each. We get $\vec{r}'_1(0) = \vec{i}$ for the first, and $\vec{r}'_2(0) = \vec{i} + 2\vec{j} + \vec{k}$ for the second. Using the geometric definition of the dot product, we have

$$(\vec{r}'_1(0)) \cdot (\vec{r}'_2(0)) = 1 = \|\vec{r}'_1(0)\| \|\vec{r}'_2(0)\| \cos(\vartheta) = \sqrt{6} \cos(\vartheta).$$

So we have $\cos(\vartheta) = 1/\sqrt{6}$, so $\vartheta = 1.15$ radians.

As with scalar derivatives, there are many rules which make derivatives of vector functions easier. They can be found on Page 862 and are generalizations of the rules (linear, scalar multiple, product etc.). There are also new rules for how derivatives behave with cross products and dot products (see rules 4 and 5). Specifically, for both, the rule is much like the standard product rule for scalar functions. We illustrate with an example.

Example 1.9. Using the rules of derivatives to find a formula for

$$\frac{d}{dt} \vec{u}(t) \cdot (\vec{v}(t) \times \vec{w}(t)).$$

We have

$$\begin{aligned} \frac{d}{dt} \vec{u}(t) \cdot (\vec{v}(t) \times \vec{w}(t)) &= \vec{u}'(t) \cdot (\vec{v}(t) \times \vec{w}(t)) + \vec{u}(t) \cdot \frac{d}{dt}(\vec{v}(t) \times \vec{w}(t)) \\ &= \vec{u}'(t) \cdot (\vec{v}(t) \times \vec{w}(t)) + \vec{u}(t) \cdot (\vec{v}'(t) \times \vec{w}(t) + \vec{v}(t) \times \vec{w}'(t)) \\ &= \vec{u}'(t) \cdot (\vec{v}(t) \times \vec{w}(t)) + \vec{u}(t) \cdot (\vec{v}'(t) \times \vec{w}(t)) + \vec{u}(t) \cdot (\vec{v}(t) \times \vec{w}'(t)) \end{aligned}$$

2. INTEGRALS OF VECTOR VALUED FUNCTIONS

The definite integral of a continuous vector valued function is defined in a very similar way to that of scalar valued function and, like the derivative, it can be calculated using components. Formally, we define the definite integral as follows.

Definition 2.1. We define the definite integral of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ from $a \leq t \leq b$ as

$$\int_a^b \vec{r}(t) dt = \lim_{n \rightarrow \infty} \sum_{i=1}^n \vec{r}(t_i^*) \Delta t = \lim_{n \rightarrow \infty} \left[\sum_{i=1}^n (f(t_i^*) \Delta t) \vec{i} + \sum_{i=1}^n (g(t_i^*) \Delta t) \vec{j} + \sum_{i=1}^n (h(t_i^*) \Delta t) \vec{k} \right]$$

where the t_i^* are distinct numbers between a and b (Riemann sums!).

With the way the definite integral is defined, it is clear how to calculate a definite integral.

Result 2.2. The definite integral of a vector valued function is calculated by taking the integrals of the components.

$$\int_a^b \vec{r}(t) dt = \int_a^b f(t) dt \vec{i} + \int_a^b g(t) dt \vec{j} + \int_a^b h(t) dt \vec{k}.$$

This result tells us that the fundamental theorem of calculus can be used to calculate definite integrals of vector valued functions. Specifically, if $\vec{R}(t)$ is an antiderivative of $\vec{r}(t)$, then

$$\int_a^b \vec{r}(t) dt = \vec{R}(b) - \vec{R}(a).$$

We finish with an example.

Example 2.3. Find $\vec{r}(t)$ if $\vec{r}'(t) = \sin(t)\vec{i} - \cos(t)\vec{j} + 3t^2\vec{k}$ and $\vec{r}(0) = \vec{0}$.

Integrating, we have

$$\vec{r}(t) = -\cos(t)\vec{i} - \sin(t)\vec{j} + t^3\vec{k} + \vec{C}$$

where \vec{C} is some constant vector. Evaluating, we have $\vec{r}(0) = -1\vec{i} + \vec{C} = \vec{0}$, so $\vec{C} = \vec{i}$ giving

$$\vec{r}(t) = (-\cos(t) + 1)\vec{i} - \sin(t)\vec{j} + t^3\vec{k}.$$