Section 14.3
Arc Length and Curvature

"Calculus on Curves in Space"

In this section, we lay the foundations for describing the movement of an object in space.

1. Vector Function Basics

In Calc 2, a formula for arc length in terms of parametric equations (in 2-space) was determined. A similar formula holds for 3-space.

Result 1.1. If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ defines a smooth curve C and $f'(t)$, $g'(t)$ and $h'(t)$ are all continuous, then the arc length along the portion of the curve with $a \leqslant t \leqslant b$ (provided it is traversed only is) is given by the formula

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$

or

$$L = \int_a^b ||\vec{r}'(t)||dt.$$

The formula is straight forward to work with:

Example 1.2. Find the length of

$$\vec{r}(t) = \vec{i} + t^2\vec{j} + t^3\vec{k}$$

for $0 \leqslant t \leqslant 1$.

This is straight forward calculations:

$$L = \int_0^1 \sqrt{0^2 + 4t^2 + 9t^4}dt = \int_0^1 2t\sqrt{4 + 9t^2}dt.$$

Substituting $u = 4 + 9t^2$, we have $du/dt = 18t$, so $dt = du/18t$, and when $t = 0$, $u = 4$, and $t = 1$ gives $u = 13$, so

$$\int_0^1 2t\sqrt{4 + 9t^2}dt = \int_4^{13} 2t\sqrt{u}\frac{1}{18t}du = \frac{1}{9}\int_4^{13} \sqrt{u}du = \frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{13}$$

$$= \frac{2}{27}\left[13^{\frac{3}{2}} - 4^{\frac{3}{2}}\right].$$

Often we are not interested in a specific distance a particle has traveled, but rather a formula to determine how far a particle has traveled in terms of some variable (this is useful in things like airline flight or space travel), so we need to determine a way to do this. Suppose $\vec{r}(u)$ is a vector function for a curve C which traverses C only once for $a \leqslant u \leqslant t$

(notice we have changed the parameter to u and we are using t as a "time variable"). We define the arclength function s for C by

$$s(t) = \int_a^t ||\vec{r}'(u)||du = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}du.$$

Notice that the integral defines a function in the variable t. Also note that by the fundamental Theorem of Calculus, this implies

$$\frac{ds}{dt} = ||\vec{r}'(t)||.$$

The function $s(t)$ measures the distance from the point $\vec{r}(a)$ to the point $\vec{r}(t)$, so as soon as we plug in a value for t, it will provide us with this numerical distance. We illustrate.

Example 1.3. Find the distance formula for $\vec{r}(u) = 2u\vec{i} + (1-3u)\vec{j} + (5+4u)\vec{k}$ from $u = 0$. Use the formula to determine the distance traveled after 2 seconds, and how long it takes for the particle to travel 8 units.

The formula is straight forward:

$$s(t) = \int_0^t \sqrt{2^2 + (-3)^2 + 4^2}du = \sqrt{29}t$$

This means after 2 seconds, the particle will have traveled $2\sqrt{29}$ units. In order to travel 8 units, we need $\sqrt{29}t = 8$, or $t = \sqrt{29}/8 \sim .672$ seconds.

Notice that in the last example, we obtained a formula for s in terms of t. In particular, we could solve this formula for t in terms of s, so $t = s/\sqrt{29}$, and then subsitute into the original parameterization for $\vec{r}$ and thus we would have a parameterization for $\vec{r}$ in terms of s instead of t. We call such a parametrization a **parameterization with respect to arclength**. This is often a preferred method of parameterization since it depends only upon the curve itself (or the length of the curve), and not a particular coordinate system. In order to determine parameterization with respect to arclength of a curve with vector equation $\vec{r}(t)$, we do the following:

(i) Solve the distance formula

$$s(t) = \int_a^t ||\vec{r}'(u)||^2du$$

for the parameterization you are given. You should have a function of s in terms of t.

(ii) Can you solve the function $s(t)$ for t? If not, you cannot reparameterize this way. Otherwise, solve for t (so t will be a function of s, $t(s)$).

(*iii*) Substitute the function $t(s)$ in for u (and rename s as u) in the original parametrization - this is now a parameterization with respect to arclength.

Example 1.4. Find a parameterization with respect to arclength for $\vec{r}(u) = 2u\vec{i} + (1-3u)\vec{j} + (5+4u)\vec{k}$ from $u = 0$.

Recall, we have

$$s(t) = \int_0^t \sqrt{2^2 + (-3)^2 + 4^2}\,du = \sqrt{29}t$$

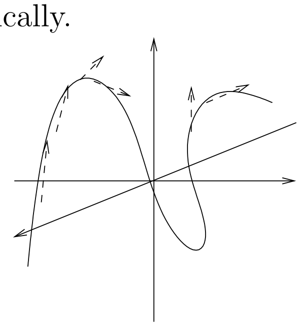
so $t = s/\sqrt{29}$. Therefore,

$$\vec{r}(t) = \frac{2u}{\sqrt{29}}\vec{i} + \frac{(1-3u)}{\sqrt{29}}\vec{j} + \frac{(5+4u)}{\sqrt{29}}\vec{k}$$

is a parameterization with respect to arclength.

2. Curvature

Recall that if C is a smooth curve defined by the vector function $\vec{r}(t)$, and $\vec{r}'(t) \neq \vec{0}$, then the unit tangent vector is given by $\vec{T}(t) = \vec{r}(t)/||\vec{r}'(t)||$ which indicates the direction of the curve. Since $\vec{T}(t)$ provides the direction of $\vec{r}(t)$, the rate of change of T with respect to s, the distance function, measures how quickly the direction of C is changing (see figure below) - when $||d\vec{T}/ds||$ is large, it means the direction of C is changing quickly over distance, and when it is small, it means it is not changing dramatically.



This motivates the following definition:

Definition 2.1. The curvature of a curve is

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

where $\vec{T}$ is the unit tangent vector with a a parameterization with respect to arclength s.

By the way we have defined κ, it seems difficult to calculate - first we need to determine the parameterization with respect to arclength of $\vec{T}$ (which as we saw previously, is not easy). However, by using the chain rule, we can avoid doing this.

Result 2.2. $\kappa(t)$ can be calculated using the following formula:

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}.$$

Observe that this completely avoids having to find s. We illustrate.

Example 2.3. Find the curvature of $\vec{r}(t) = 3t\vec{i} + 4\sin(t)\vec{j} + 4\cos(t)\vec{k}$.

First we find the unit normal vector,

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{3\vec{i} + 4\cos(t)\vec{j} - 4\sin(t)\vec{k}}{\sqrt{(3^2+4^2)}} = \frac{3}{5}\vec{i} + \frac{4}{5}\cos(t)\vec{j} - \frac{4}{5}\sin(t)\vec{k}.$$

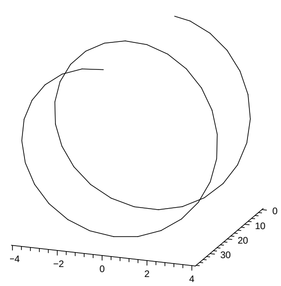
Then we have

$$\vec{T}'(t) = -\frac{4}{5}\sin(t)\vec{j} - \frac{4}{5}\cos(t)\vec{k}.$$

Then we have

$$\kappa(t) = \frac{|| - \frac{4}{5}\sin(t)\vec{j} - \frac{4}{5}\cos(t)\vec{k}||}{||3\vec{i} + 4\cos(t)\vec{j} - 4\sin(t)\vec{k}||} = \frac{\frac{4}{5}}{5} = \frac{4}{25}.$$

Notice that this is constant (it does not depend upon t) which means that the curvature is constant. This is also apparent from the graph below where we can see the tangent vectors are changing at a constant rate:



There are other ways to calculate curvature which do not rely upon finding the tangent vector and instead use a cross-product.

Result 2.4. The curvature of the curve C given by $\vec{r}(t)$ is

$$\kappa(t) = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3}.$$

Example 2.5. Find the curvature of $\vec{r}(t) = \sqrt{2}t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}$.

We shall apply the recent formula: we have $\vec{r}'(t) = \sqrt{2}\vec{i} + e^t\vec{j} - e^{-t}\vec{j}$, and $\vec{r}''(t) = e^t\vec{j} + e^{-t}\vec{k}$, so

$$\vec{r}'(t) \times \vec{r}''(t) = 2\vec{i} - \sqrt{2}e^{-t}\vec{j} + \sqrt{2}e^t\vec{k}$$

so

$$||\vec{r}'(t) \times \vec{r}''(t)|| = \sqrt{4 + 2e^{2t} + 2e^{-2t}} = \sqrt{2}\sqrt{(e^t + e^{-t})^2} = \sqrt{2}(e^t + e^{-t})$$

We also have

$$||\vec{r}'(t)||^3 = (4 + 2e^{2t} + 2e^{-2t})^{3/2} = (\sqrt{2}(e^t + e^{-t}))^3$$

so

$$\kappa(t) = \frac{1}{2(e^t + e^{-t})^2}.$$

For the special case of a plane curve $y = f(x)$, (so a 2-dimensional curve), the z-coordinate is always zero, so we can take the parameter to be x so a vector equation will be

$$\vec{r}(x) = x\vec{i} + f(x)\vec{j}.$$

Then we shall have

$$||\vec{r}'(x) \times \vec{r}''(x)|| = ||(\vec{i} + f'(x)\vec{j}) \times (f''(x)\vec{j})|| = ||f''(x)\vec{k}|| = |f''(x)|,$$

and

$$||\vec{r}'(x)||^3 = ||(\vec{i} + f'(x)\vec{j})||^3 = ||1 + (f'(x))^2||^3 = |1 + (f'(x))^2|^{3/2}.$$

Thus we have

Result 2.6. If $y = f(x)$, then the curvature at any point x is given by the formula

$$\kappa(x) = \frac{|f''(x)|}{|1 + (f'(x))^2|^{\frac{3}{2}}}$$

Example 2.7. Find the curvature of $y = x^3$ at $(1, 1)$.

We just apply the formula,

$$\kappa(1) = \frac{6}{10^{3/2}}$$

3. Normal and Binormal Vectors

We have already seen that at any point on a curve, there is a vector called the unit tangent vector which tells us the direction the curve is going. There are two other vectors closely related to this vector which also provide information about the curve C. Specifically, provided the unit tangent vector is non-zero, we can find two other vectors which are perpendicular to it and are mutually perpendicular to each other (giving something like a coordinate axis at the point). We define them as follows:

Definition 3.1. Suppose C is a curve with vector equation $\vec{r}(t)$ and let $\vec{T}(t)$ be its unit tangent vector defined as

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||^2}.$$

Then we define:

(i) The principal unit normal vector $\vec{N}(t)$ defined as

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||^2}.$$

(ii) The binormal vector $\vec{B}(t)$ defined as

$$\vec{B}(t) = \vec{T}(t) \times \vec{B}(t).$$

All three vectors are mutually perpendicular.

Calculation of these vectors though cumbersome, is fairly straight forward. We give an example.

Example 3.2. Find the vectors $\vec{T}$, $\vec{N}$ and $\vec{B}$ for $\vec{r}(t) = t^2\vec{i} + 2t^3/3\vec{j} + t\vec{k}$ at the point $(1, 2/3, 1)$.

We have

$$\vec{T}(t) = \frac{2t\vec{i} + 2t^2\vec{j} + \vec{k}}{\sqrt{(4t^4 + 4t^2 + 1}} = \frac{2t\vec{i} + 2t^2\vec{j} + \vec{k}}{\sqrt{(2t^2+1)^2}} = \frac{2t\vec{i} + 2t^2\vec{j} + \vec{k}}{(2t^2+1)}.$$

To calculate $\vec{T}'(t)$, we use the generalized product rule:

$$\vec{T}'(t) = -\frac{4t}{(2t^2+1)^2}(2t\vec{i} + 2t^2\vec{j} + \vec{k}) + \frac{1}{(2t^2+1)}(2\vec{i} + 4t\vec{j})$$

$$= -\frac{1}{(2t^2+1)^2}(8t^2\vec{i} + 8t^3\vec{j} + 4t\vec{k}) + \frac{1}{(2t^2+1)^2}((4t^2+2)\vec{i} + (8t^3+4t)\vec{j})$$

$$= \frac{1}{(2t^2+1)^2}((2-4t^2)\vec{i} + 4t\vec{j} - 4t\vec{k}) = \frac{2}{(2t^2+1)^2}((1-2t^2)\vec{i} + 2t\vec{j} - 2t\vec{k}).$$

Then we also have

$$||\vec{T}'(t)|| = \frac{2}{(2t^2+1)^2}\sqrt{(1 - 4t^2 + 4t^4 + 4t^2 + 4t^2)} = \frac{2}{(2t^2+1)}$$

so

$$\vec{N}(t) = \frac{\frac{2}{(2t^2+1)^2}((1-2t^2)\vec{i} + 2t\vec{j} - 2t\vec{k})}{\frac{2}{(2t^2+1)}} = \frac{1}{2t^2+1}((1-2t^2)\vec{i} + 2t\vec{j} - 2t\vec{k}).$$

At $(1, 2/3, 1)$ we have $t = 1$, so

$$\vec{T}(1) = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{9}} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

and

$$\vec{N}(1) = -\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k},$$

so

$$\vec{B}(1) = (\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}) \times (-\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}) = -\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

Recall that given a pair of (non-parallel) vectors and a point, there is a plane which contains both vectors and the point. The planes associated to the vectors we have introduced above provide useful information about a curve C with vector $\vec{r}(t)$, so we introduce some terminology for these planes:

Definition 3.3. Suppose that $\vec{r}(t)$ is a vector equation for a curve C. For a fixed value, $t = a$, we define the following:

(i) The plane containing the point $\vec{r}(a)$ and vectors $\vec{N}$ and $\vec{B}$ is called the **normal plane** to C at $P = \vec{r}(a)$. It contains all vectors orthogonal to C at $t = a$.

(ii) The plane containing the point $\vec{r}(a)$ and vectors $\vec{N}$ and $\vec{T}$ is called the **osculating plane** to C at $P = \vec{r}(a)$. It is the plane which $\vec{C}$ most closely lies in at P.

Example 3.4. Find the equations for the normal plane and the osculating plane to $\vec{r}(t) = t^2\vec{i} + 2t^3/3\vec{j} + t\vec{k}$ at the point $(1, 2/3, 1)$.

We have already determined the three required vectors:

$$\vec{T}(1) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

and

$$\vec{N}(1) = -\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k},$$

and

$$\vec{B}(1) = -\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

The normal plane contains $\vec{B}$ and $\vec{N}$, so a normal vector to the plane will be $\vec{T}$ (which is perpendicular to both $\vec{B}$ and $\vec{N}$). This an equation will be

$$\frac{2}{3}(x-1) + \frac{2}{3}(y - \frac{2}{3}) + \frac{1}{3}(z-1) = 0.$$

The osculating plane will have $\vec{B}$ as a normal vector, so will have equation

$$-\frac{2}{3}(x-1) + \frac{1}{3}(y - \frac{2}{3}) + \frac{2}{3}(z-1) = 0.$$