Section 14.4
Motion in Space: Velocity and Acceleration

"Applications of vector functions and vector derivatives"

In Calc 1 and 2, we saw that derivatives and integrals were closely related to the concept of speed, distance traveled and acceleration. In this section, we shall generalize these ideas to vectors, tracking the motion of a body in 3-dimensional space (rather than the rather fake 2-d space developed in Calc 2).

1. Velocity, Acceleration and Force Vectors

Suppose that $\vec{r}(t)$ is the position function for a particle P traveling through space. Then we define its velocity vector and acceleration vector as follows:

Definition 1.1. The velocity (the rate of change of position with respect to time) of P at time t is defined to be

$$\vec{v}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \vec{r}'(t)$$

and the acceleration (the rate of change of velocity with respect to time) of P at time t is defined to be

$$\vec{a}(t) = \lim_{h \to 0} \frac{\vec{v}(t+h) - \vec{v}(t)}{h} = \lim_{h \to 0} \frac{\vec{r}'(t+h) - \vec{r}'(t)}{h} = \vec{r}''(t).$$

We define the speed of P at time t to be $||\vec{r}'(t)||$ (this is the physical rate of change of distance with respect to time).

Calculation of these values are straight forward.

Example 1.2. Find the velocity, acceleration and speed of a particle whose position in space is given by

$$\vec{r}(t) = \sin(t)\vec{i} + t\vec{j} + \cos(t)\vec{k}.$$

We have $\vec{r}'(t) = \cos(t)\vec{i} + \vec{j} - \sin(t)\vec{k}$, $\vec{r}''(t) = -\sin(t)\vec{i} - \cos(t)\vec{k}$, and $||\vec{r}'(t)|| = \sqrt{(\cos^2(t) + 1 + \sin^2(t))} = \sqrt{2}$.

If instead we are given the acceleration, we can easily find the velocity and distance functions using integration. Another vector which can arise in applications in the sciences is the force vector. Force is related to acceleration and mass using Newtons second law. Specifically, it says the following:

Definition 1.3. If a force $\vec{F}(t)$ acts on an object of mass m producing an acceleration $\vec{a}(t)$, then

$$\vec{F}(t) = m\vec{a}(t).$$

The following example considers the action of the force of gravity on the path of a projectile.

Example 1.4. A projectile of mass $m = 50N$ is fired with an initial speed of 500 m/s and angle of elevation 30 degrees. Assuming air resistance is negligible (so the only external force is due to gravity), find the following:

- (i) the range of the projectile
- (ii) the maximum height reached
- (iii) the speed at impact

We set up the axis so the projectile is fired from the origin in the direction of the positive x-axis (so the $\vec{j}$ component of the velocity, acceleration and position vectors is 0). We know the following:

- (i) If $\vec{v}(t) = f(t)\vec{i} + g(t)\vec{k}$ is the velocity vector, we know $||\vec{v}(0)|| = \sqrt{f(0)^2 + g(0)^2} = 500$.
- (ii) The initial direction of the projectile is in the direction of the unit vector $(\sqrt{3}\vec{i} + \vec{k})/2$ (since it is fired at a 30 degree inclination). Therefore, $\vec{v}(0) = 250(\sqrt{3}\vec{i} + \vec{j})$ is the initial velocity vector.
- (iii) The only acting force on the particle is gravitation. Note that acceleration due to gravitation is constant with respect to time independent of mass, so the acceleration vector for the projectile will be $\vec{a}(t) = -9.8\vec{k}$. Alternatively, we could have used Newtons second law to derive this. Specifically, the force acting om the particle will have vector $\vec{F}(t) = -9.8 * 50\vec{k} = -490\vec{k}$. Using Newtons second law, we can derive acceleration as $\vec{F}(t)/50 = \vec{a}(t) = -9.8\vec{k}$ (this is just the downward acceleration

It follows that

$$\vec{v}(t) = a\vec{i} - (9.8t + b)\vec{k}.$$

Setting $t = 0$, we have $\vec{v}(0) = a\vec{i} + b\vec{k} = 250(\sqrt{3}\vec{i} + \vec{j})$ and so

$$\vec{v}(t) = 250\sqrt{3}\vec{i} + (250 - 9.8t)\vec{k}.$$

Since the initial position is $\vec{0}$, we have

$$\vec{s}(t) = 250\sqrt{3}t\vec{i} + (250t - 4.9t^2)\vec{k}.$$

We can now answer the questions:

- (i) The range of the projectile is the horizontal distance it will travel. To calculate this, we need to know when it hits the ground. This occurs when $(250t - 4.9t^2) = 0$ or at the times $t = 0$ or $t = 51$ seconds. At the latter of these times, the horizontal distance will be $250\sqrt{3} * 51 = 22,083$m (or 22km).

(*ii*) The maximum height reached will be halfway through the motion, when $t = 25.5$. We could also find this by differentiating the height function to find where its mins and maxes are. Plugging this into the height function, we get 3188m or about 3km.

(*iii*) Finally, the velocity on impact will be $250\sqrt{3}\vec{i} - 249.9\vec{k}$, so the speed will be $\sqrt{(250\sqrt{3})^2 + (249.9)^2} = 499.5$m/s.

2. Tangential and Normal Components of Acceleration

When studying the motion of an object, it is often useful to resolve the acceleration vector into two components - one in the direction of the tangent (so the amount we are accelerating in the direction we are traveling) and the normal (the amount we are accelerating in a perpendicular direction). If $v = ||\vec{v}(t)||$ is the speed of an object whose position is given by the vector function $\vec{r}(t)$, then $\vec{a}$ can be resolved in terms of the normal and tangent vectors $\vec{T}$ and $\vec{N}$. Specifically, we have

$$\vec{a}(t) = v'\vec{T} + \kappa v^2 \vec{N}.$$

Instead of calculating v, we can write these equations in terms of $\vec{r}$ only,

$$\vec{a}(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{||\vec{r}'(t)||}\vec{T}(t) + \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||}\vec{N}(t).$$

We finish with an example.

Example 2.1. Find the tangential and normal components of the acceleration vector of $\vec{r}(t) = (3t - t^3)\vec{i} + 3t^2\vec{j}$.

This is straight forward (but very frustrating) calculation, we need to determine each of the necessary components: $\vec{r}'(t) = (3 - 3t^2)\vec{i} + 6t\vec{j}$, $\vec{r}''(t) = -6t\vec{i} + 6\vec{j}$,

$$||\vec{r}'(t)|| = \sqrt{(3 - 3t^2)^2 + (6t)^2} = \sqrt{9 - 18t^2 + 9t^4 + 36t^2} = \sqrt{9 + 18t^2 + 9t^4}$$

$$= \sqrt{9(1 + t^2)^2} = 3(1 + t^2),$$

$\vec{r}'(t) \cdot \vec{r}''(t) = -18t + 18t^3 + 36t = 18t^3 + 18t$ and

$$\vec{r}'(t) \times \vec{r}''(t) = (18t^2 + 18t)\vec{k}.$$

Since

$$\vec{a}(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{||\vec{r}'(t)||}\vec{T}(t) + \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||}\vec{N}(t),$$

so the first component is

$$\frac{\vec{r}'(t) \cdot \vec{r}''(t)}{||\vec{r}'(t)||} = \frac{18t^3 + 18t}{3(t^2 + 1)} = 6t$$

and the second component is

$$\frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||} = \frac{18t^2 + 18t}{3(t^2 + 1)} = 6.$$