Section 15.1
Functions of Several Variables

"Defining and Graphing Functions of more than one Variable" Except for a couple of special examples (cylindrical surfaces, planes), we have not yet really touched upon the idea of functions of more than one variable (instead we have focused on curves in 3-space). In this section, we shall consider the basic properties of a function of two variables.

1. Functions of Two Variables - Basic Definitions

We start with a formal definition of a function of two variables.

Definition 1.1. A function of two variables f is a rule that assigns to an ordered pair of real numbers (x, y) a unique real number denoted by $f(x, y)$. The domain of $f(x, y)$ is the set of all ordered pairs (x, y) for which f can be evaluated, and the range is all the values that f takes.

There are many different ways to represent a function of two variables. Just like single variable functions, one way is to list the entries in a table. The only problem with this method is that just a couple of data points requires a large table. We illustrate.

Example 1.2. Market research suggests that the number of people who use the train to commute between Salem and Portland every morning is a function of the cost of a ticket and the time it takes to get from Salem to Portland, so $N(T, P)$ where $N =$ the number of people, T is time and P is price. Research indicates the following results:

	Cost P ($)				
Time T (Hrs)	5	10	15	20	25
1	35	33	28	20	15
1.5	34	30	27	20	12
2	20	15	9	7	1
2.5	12	10	7	1	0

(i) How many people take the train if it take 1(1/2) hours and costs $ 15?

We have $N(1.5, 15) = 27$

(ii) In order to maximize revenue, what is the best time and price to choose?

The revenue will be equal to the number of people who take the train multiplied by the cost for a ticket. Therefore, we can construct a new function for revenue depending upon cost P

and time T in hours by simply multiplying each column in the previous table by the price of the ticket:

Time (Hrs)	Cost (\$) 5	10	15	20	25
1	175	330	420	400	375
1.5	170	300	405	400	300
2	100	150	135	140	25
2.5	60	100	105	20	0

We can see from the table that the most revenue is gathered when $P = \$15$ and $T = 1$.

As the last example suggests, using tables to represent functions of two variables is inefficient - in general, we need a lot more information to make even simple conclusions about a function. Therefore, as with single variable, we usually use graphs and equations to represent functions and study them. For algebraic representations, many of the results regarding functions will be identical to the single variable case. With graphs however, it is more complicated. We explore some algebraic expressions first.

Example 1.3. Find the domain and range of $f(x, y) = \sqrt{16 - x^2 - y^2}$ and try to sketch a graph of $f(x, y)$.

The domain of f is all the allowable input values. Since we cannot take the square root of a negative number, we must have $16 - x^2 - y^2 \geqslant 0$, or $16 \geqslant x^2 + y^2$. This means the domain of $f(x, y)$ is all points in the circle of radius 4 centered at the origin in the xy-plane.

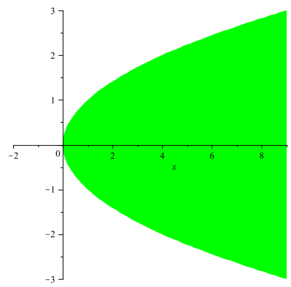
Since $x^2 + y^2 \geqslant 0$, it follows that $16 - x^2 - y^2 \leqslant 16$, so $\sqrt{16 - x^2 - y^2} \leqslant 4$. $f(x, y)$ clearly takes every value between 0 and 4, so the range is $0 \leqslant f(x, y) \leqslant 4$.

Since the graph of $f(x, y)$ is the set of all points $(x, y, f(x, y))$ (so the function value is the z-value), we need to determine what this looks like. However, if we set $z = f(x, y) = \sqrt{16 - x^2 - y^2}$, we have $z^2 = 16 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 16$, a sphere of radius 4 centered at the origin. Observe however that it is only the upper half sphere since $z \geqslant 0$.

As the last example suggested, finding the graph of a function is fairly complicated. We consider this problem in more detail in the next section. Before that, we consider one more example.

Example 1.4. Sketch the domain of the function $f(x, y) = \ln(x - y^2)$.

The domain will of $\ln(x)$ is all positive real numbers, so we need to determine the region in the xy-plane such that $y^2 - x > 0$, or equivalently, $x > y^2$. This is true provided $y < \sqrt{x}$ and $y > -\sqrt{x}$ i.e. this will be the region bounded between the two functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ as illustrated below:



2. The Graph of a Function of Two Variables

Just like with single variable, the graph of a function of two variables is defined to be the set of all points $(x, y, f(x, y))$ such that (x, y) is in the domain of f and the z-value of the value of f at (x, y). There are different techniques to help draw graphs which we shall consider:

- (i) Technology (calculators, computers) - useful, but sometimes misleading
- (ii) Familiar graphs - linear functions, spheres, cylindrical surfaces
- (iii) Contour diagrams or level curves

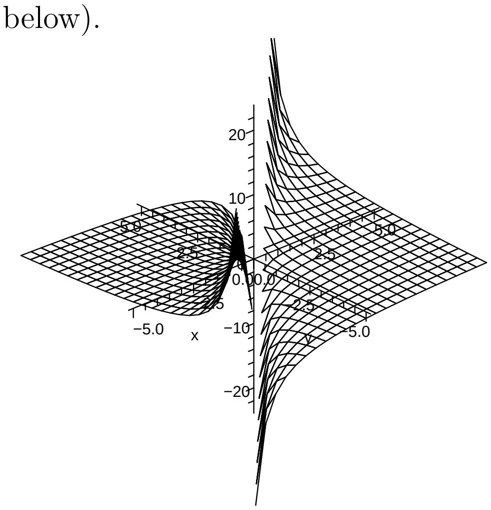
We shall look at examples of each of these ideas.

2.1. **Technology.**

Example 2.1. Use technology to draw the graph of

$$f(x, y) = \frac{x + y}{x - y}.$$

The TI-89 has the capability to sketch 3D graphs. However, looking at the quality of this graph, we see that the TI-89 is fairly limited with its uses! Instead one could use a computer algebra system like Maple (see illustration below).

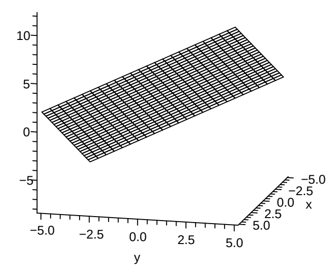


2.2. **Familiar Graphs.** Of course, we do not want to have to rely on technology to draw graphs, and even when we do, we need to know at which points the graph is interesting in order to even choose a viewing window. We have already looked at certain special graphs which are graphs of functions. We illustrate.

Example 2.2. The following are equations for surfaces you should recognize. Use this knowledge to sketch a graph.

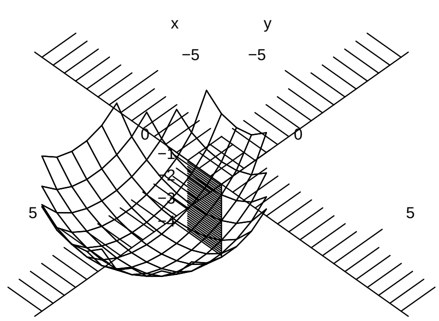
(*i*) $f(x, y) = 2 + y - x$

This function is a **linear function** in x and y (the powers of x and y are at most 1). If we replace $f(x, y)$ by z, we have $z = 2 + y - x$, which is a linear equation in x, y and z. In particular, its graph will be a plane. When $z = 0$, this plane will have equation $y = x - 2$, so it passes through this line. When $z = 2$, this has equation $y = x$, so it also passes through this line. This is enough information to sketch the plane:



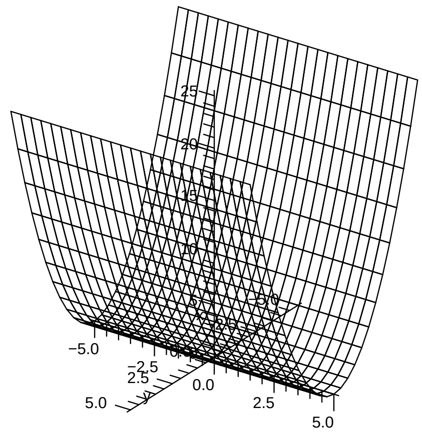
(*ii*) $f(x, y) = -\sqrt{16 - (x - 1)^2 - (y + 2)^2}$

This is similar to the first example we did - it is the bottom half of a sphere of radius 4 centered at the point $(1, -2, 0)$



(iii) $f(x, y) = x^2$

This is a cylindrical surface - it extends in the y-direction without changing z-value. Therefore, we just sketch the curve in the xz-plane and then draw out in the y-direction.



2.3. **Level Curves.** Of course, there are many more functions which are not like the few that we recognize. In order to take care of these, we can use a technique used in cartography to represent hills on a map - the idea of contour diagrams or level sets. We construct a contour diagram and use it as follows:

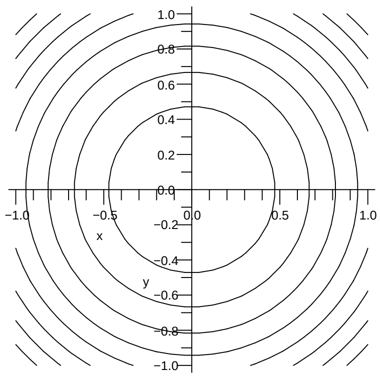
- For each fixed value $z = c$, we can set $c = f(x, y)$ - this is an equation in x and y whose graph can be drawn in a plane. The graph of this equation in x and y is called the contour (or **level set** or **level curve**) of $f(x, y)$ at $z = c$. We sketch it the xy-plane and label it with the number c.
- We can allow c to vary and draw lots of different contours on the same plane labeling each. This is called a contour diagram for $f(x, y)$ and describes the intersection of $f(x, y)$ with different planes parallel to the xy-plane.
- Usually we choose the contours to be equally spaced - that way we can guess the shape of the surface from the contour diagram - if the contours are close together, the graph is steep, if they are far apart, the slope is gentle.

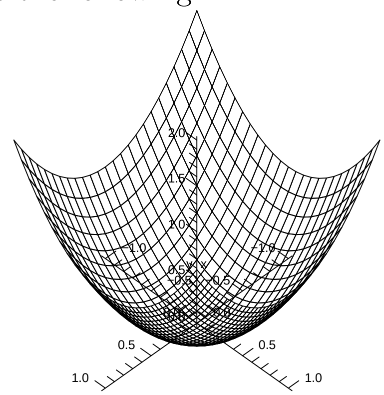
We illustrate with some examples.

Example 2.3. Use contour diagrams to sketch graphs for the following functions.

(i) $f(x, y) = x^2 + y^2$

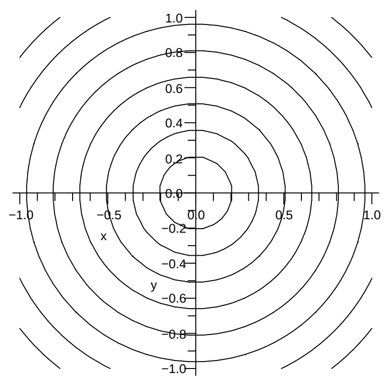
For a fixed c, the contour is a circle of radius $\sqrt{c}$. This means the contour diagram will consist of circles with increasing radius (as c increases), but the rate of change of the radius is decreasing (it is getting larger at a smaller rate). This suggests a contour diagram and graph like the following:

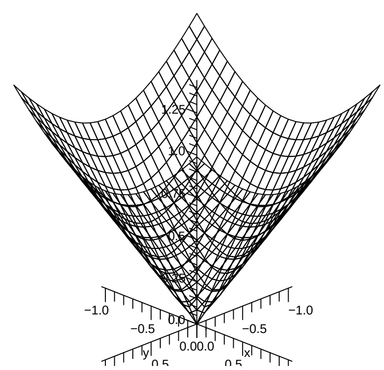




(*ii*) $f(x,y) = \sqrt{x^2 + y^2}$

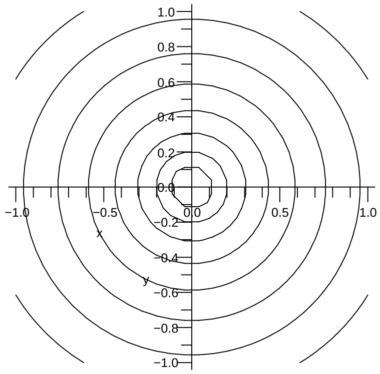
For a fixed c, the contour is a circle of radius c. This means the contour diagram will consist of circles with increasing radius (as c increases), and the rate of change of the radius is constant (it is getting larger at a constant rate). This suggests a contour diagram and graph like the following:

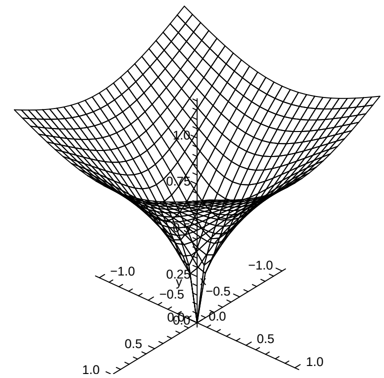




(*iii*) $f(x,y) = (x^2 + y^2)^{(1/4)}$

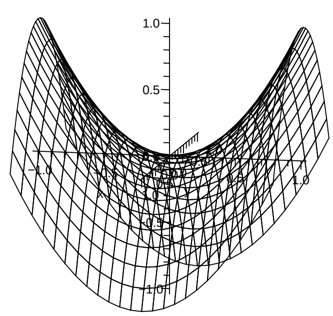
For a fixed c, the contour is a circle of radius c^4. This means the contour diagram will consist of circles with increasing radius (as c increases), but the rate of change of the radius is increasing (it is getting larger at a faster rate). This suggests a contour diagram and graph like the following:

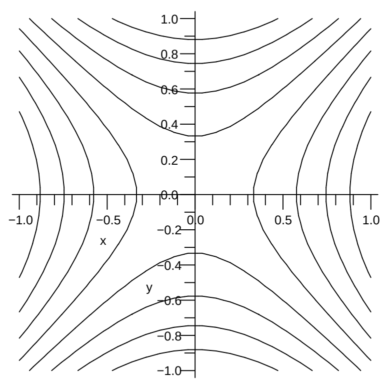




Observe that all these contour diagrams were basically the same - circles of increasing radius centered at the origin. The major difference is the spacing of these circles.

Example 2.4. Sketch a contour map of the function $f(x, y) = x^2 - y^2$ and use it to sketch its graph.

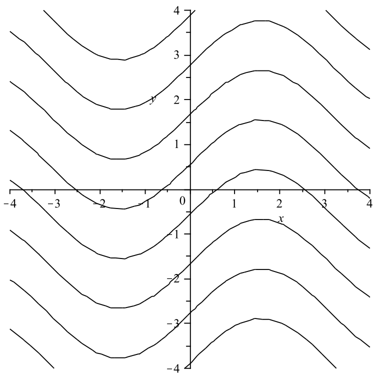




The general shape of this graph is called a saddle - there is a maximum at the origin in one direction, and a min at the other. It is shapes like this which will cause problems when we attempt to find maximums and minimums of functions of more than one variable.

Example 2.5. Sketch the contour diagram for $f(x, y) = y - \sin(x)$. If you walk along the line $x = \pi/2$, are you walking along a ridge or in a valley?

For a fixed value $z = c$, we have $y = c + \sin(x)$, so it follows that the contours of $f(x, y)$ will be vertical shifts of the function $g(x) = \sin(x)$ with c increasing as we move up vertically in the y-direction. We llustrate:



Suppose that we are stood on the line $x = \pi/2$ at the first contour above the x-axis. Then the z-values immediately to our left and right are larger than the z value at which we are stood since they are above the contour we are stood on (recall that the z-values of the contours are increasing as we move vertically upward in the y-direction). It follows that we must be walking in a valley (even though it looks like we are walking along a ridge!)

3. Functions of Three or more Variables

When considering functions of more than two variables, the level sets can still be used to get an idea of what the graph looks like, but the level sets will not be curves - they will be surfaces for functions of three variables, and then higher dimensional objects for higher dimensions. Since we are mainly interested in functions of two variables, we shall not examine other functions in too much detail. We illustrate with one example.

Example 3.1. What are the level sets of the function $f(x, y) = x^2 + y^2 + z^2$?
For a fixed positive c, the graph of $c = x^2 + y^2 + z^2$ is a sphere of radius $\sqrt{c}$ centered at the origin. This means for a fixed c, the level sets are spheres of radius $\sqrt{c}$ centered at the origin.