Section 15.2
Limits and Continuity

"Generalizing Ideas from Single Variable Calculus"

The ideas of limits and continuity were critical when defining the derivative in single variable calculus -

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We shall generalize these ideas to functions of more than one variable.

1. Limits of Functions of Two Variables

Recall that the naive idea of the limit of a function $f(x)$ at a point $x = a$ is the following: it is the value $f(x)$ tends towards as x gets close to a, where by "close", we mean $|x - a|$ is sufficiently small. This causes a problem when defining a limit of a function of two variables - the value $|(x, y) - (a, b)|$ makes no sense. However, the term "close" refers to distance, and we have a general formula for distance between points in the plane. Specifically, two points (x, y) and (a, b) are close provided $\sqrt{(x-a)^2 + (y-b)^2}$ is sufficiently small. We can use this definition to define the limit at a point of a function of two variables.

Definition 1.1. Let f be a function of two variables whose domain includes points arbitrarily close to (a, b) (though not necessarily (a, b)). Then we say the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and write

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever (x, y) is in the domain of f and

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Visually, since the domain of $f(x, y)$ is a region in 2-space, this means that if we choose a point (x, y) within a ball of radius $\delta > 0$ centered at (a, b), then the value $|f(a, b) - f(x, y)| < \varepsilon$. The major difference between single variable and multivariable is that there are many different ways to approach a point in 2-space (as opposed to just two ways in 1-space along the real line). In particular, the limit has to be independent of the path taken to get to a point - this is sometimes a test to check whether or not a limit exists at a point.

Result 1.2. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2 where $L_1 \neq L_2$, then the limit $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist.

In general, showing that a limit exists at a point is much more difficult that showing it does not exist at a point. We illustrate with some examples.

Example 1.3. Find the following limits, if they exist, or explain why they do not.

(i)

$$\lim_{x \to (0,0)} \frac{x^2}{x^2 + y^2}$$

We can approach $(0,0)$ along the x-axis (in which case $y = 0$). This gives

$$\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2} = 1.$$

However, if we approach on the y-axis, so $x = 0$, we have

$$\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \frac{0}{0 + y^2} = 0.$$

Thus the limit does not exist.

(ii)

$$\lim_{x \to (0,0)} \frac{xy^4}{x^2 + y^8}$$

Along the x-axis, we have

$$\lim_{x \to (0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{x \to (0,0)} \frac{0}{y^8} = 0.$$

However, approaching along the curve $x = y^4$, we have

$$\lim_{x \to (0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{x \to (0,0)} \frac{y^8}{y^8 + y^8} = \frac{1}{2},$$

so the limit does not exist.

The definition of the limit we have given for functions of two variables in fact holds for functions of any number of variables. We illustrate with an example.

Example 1.4. Determine whether the following limit exists:

$$\lim_{x \to (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$$

The idea is the same as with the previous examples - we just need to determine if the limit is independent of path. Observe that moving along the path $z = 0$ and $y = x$, we have

$$\lim_{x \to (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} = \lim_{x \to (0,0,0)} \frac{x^2}{x^2 + x^2} = \lim_{x \to (0,0,0)} \frac{x^2}{2x^2} = \frac{1}{2}.$$

However, if we move along the path $y = z^2$ and $x = z^2$, we have

$$\lim_{x\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} = \lim_{x\to(0,0,0)} \frac{z^4 + z^4 + z^4}{z^4 + z^4 + z^4} = 1,$$

so the limit does not exist.

2. Continuity

The definition of continuity for a function of two variables is a direct generalization of continuity for a function of a single variable.

Definition 2.1. A function $f(x, y)$ of two variables is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We say it is a continuous function if it is continuous at every point of its domain. (For a function of three variables, we say $f(x, y, z)$ of two variables is continuous at (a, b, c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c).$$

We say it is a continuous function if it is continuous at every point of its domain.)

Intuitively, this simply means that as we get close to a point, there are no jumps, holes or infinite oscilllations - the graph is “smooth”. Examples of continuous functions in more than one variable are: polynomials (expressions involving positive integer powers of variables) are continuous everywhere; rational functions (quotients of polynomials) are continuous everywhere they are defined; the composition of two functions are continuous where ever the original functions are. We illustrate with some examples.

Example 2.2. Determine the sets of points where the following functions are continuous:

(i)

$$f(x, y) = \arctan(x + \sqrt{y})$$

This is a composition of $x + \sqrt{y}$ and $\arctan s$. The function $\arctan(s)$ is continuous everywhere, so $f(x, y)$ will be continuous where ever $x + \sqrt{y}$ is. This is continuous and defined provided $y \geqslant 0$, so this function is continuous on the set

$$D = \{(x, y)|y \geqslant 0\}$$

which is the half plane with positive y.

(ii)

$$f(x, y, z) = \frac{\sqrt{y}}{x^2 - y^2 + z^2}$$

This function is continuous provided it is defined. Its domain of definition must have $y > 0$ and and $x^2 + z^2 \geqslant y^2$, so the $f(x, y, z)$ will be continuous on

$$D = \{(x, y, z)|y > 0, x^2 + y^2 \geqslant z^2\}$$

which is the exterior of a cone with base at the origin, centered on the y-axis with $y > 0$.

(iii) Determine whether the function

$$\begin{cases} \frac{x^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}$$

Observe that approaching $(0, 0)$ along the x-axis gives

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1.$$

However, if we approach on the y-axis, so $x = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{0 + y^2} = 0.$$

Thus the limit does not exist at this point and therefore the function cannot be continuous at this point.

We finish with an example showing how contour diagrams can be used to evaluate limits and continuity..

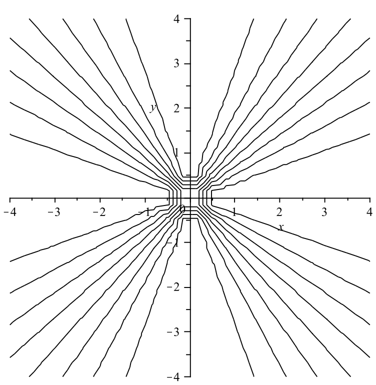
Example 2.3. Use contour diagrams of

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

to show that

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist.



Observe that close to $(0, 0)$, lots of different contours seem to come together. This means that the limit cannot possibly exists at $(0, 0)$, since $f(x, y)$ takes lots of different z values close to $(0, 0)$ dependent upon the direction of approach.