Section 15.3
Partial Derivatives

"Differentiating Functions of more than one Variable"

1. Basic Definitions

In single variable calculus, the derivative is defined to be the instantaneous rate of change of a function $f(x)$. In multivariable, this definition no longer makes sense because there are many different directions in which one could move, so the rate of change will depend not only upon the point we are at, but also the direction we choose to move. We illustrate with an example.

Example 1.1. A sheet of unevenly heated metal lies in the xy-plane with the lower left corner at the origin. The temperature at any point of the sheet is a function of x and y, $T(x, y)$. After taking some measurements, you gather the following information:

3	85	90	110	135	155	180
2	100	110	120	145	190	170
1	125	128	135	160	175	160
0	120	135	155	160	160	150
y/x	0	1	2	3	4	5

If we are stood at the point $(2, 1)$ in the xy-plane, the temperature changes depending upon the direction we choose to mover. If we fix the y-value at 1 and move in the positive x-direction, then the function becomes a function of a single variable, so we can consider its rate of change in the x-direction. Specifically, we have

$$T'(x, 1) \sim \frac{T(3, 1) - T(2, 1)}{1} = \frac{160 - 135}{1} = 25,$$

so the rate of change of temperature in the x-direction is approximately 25 degrees per unit moved. This means that the temperature is increasing in the x-direction quite quickly.
Likewise, in the y-direction we can consider the rate of change like with a single variable function. Specifically, fixing $x = 2$, we have

$$T'(2, y) \sim \frac{T(2, 2) - T(2, 1)}{1} = \frac{120 - 135}{1} = -15,$$

so the rate of change is negative in the y-direction (meaning the temperature is dropping in the y-direction).

In order to define derivatives for functions of more than one variable, we can generalize the ideas from this example. That is, we can consider rates of change of a function in the directions of the different coordinate axis by keeping all other variables fixed and treating the function like a function of a single variable. We formalize the definition:

Definition 1.2. Suppose $f(x, y)$ is a function of two variables. Then we define its **partial derivative** to be the functions f_x and f_y defined by

$$f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

There are many different notations used to denote partial derivatives. Some of the more common are:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_1 = D_1 f = D_x f$$

and likewise for y.

Since the definition for partial derivatives is a direction generalization of regular single variable derivatives, calculation is almost identical. Specifically, we have:

Result 1.3. To find the partial derivatives of $f(x, y)$, we do the following:

(i) To find f_x, treat y like a constant and differentiate with respect to x.
(ii) To find f_y, treat x like a constant and differentiate with respect to y.

Example 1.4. Let $f(x, y) = \sin(x) + y^2 - xy$.

(i) Find the partial derivatives of $f(x, y)$.

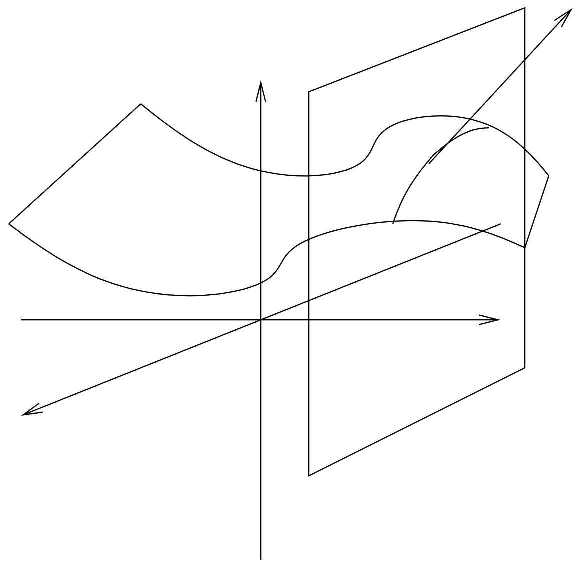
This is straight forward - $f_x = \cos(x) - y$ and $f_y = 2y - x$.

(ii) Find $f_x(0, 2)$.

We know $f_x = \cos(x) - y$, so $f_x(0, 2) = 1 - 2 = -2$.

2. Interpreting Partial Derivatives

As with single variable calculus, partial derivatives have a meaningful geometric interpretation. Specifically, the value of $f_x(a, b)$ is the slope of the function $f(x, y)$ at the point (a, b) in the direction of the unit vector $\vec{i}$. Likewise, $f_y(a, b)$ is the slope of the function $f(x, y)$ at the point (a, b) in the direction of the unit vector $\vec{j}$.
Alternatively, we can think of $f_x(a, b)$ as the slope of the curve at the intersection of the graph of $z = f(x, y)$ with the plane $y = b$ at the point $x = a$ (see illustration below). Likewise, we can think of $f_y(a, b)$ as the slope of the curve at the intersection of the graph of $z = f(x, y)$ with the plane $x = a$ at the point $y = b$.



Example 2.1. Find the slopes of the function $f(x, y) = x^2 + y^2$ at the point $(2, 3)$ in the y and x directions.

$f_x = 2x$, so $f_x(2, 3) = 4$, and $f_y = 2y$, so $f_y(2, 3) = 6$.

Example 2.2. If

$$f(x, y) = \frac{\sin(x)}{x^2 + y}$$

find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

We have

$$\frac{\partial f}{\partial x} = \frac{\cos(x)(x^2 + y) - 2x\sin(x)}{(x^2 + y)^2} \text{ and } \frac{\partial f}{\partial y} = \frac{-\sin(x)}{(x^2 + y)^2}$$

3. Functions of more than two variables

Partial derivatives can be define for functions of lots of variables in exactly the same way as we defined them for functions of two variables. Specifically, if $y = f(x_1, \ldots, x_n)$ is a function in n variables, we define the partial derivative f_{x_i} to be the slope in the direction of the x_i axis. It is calculated in exactly the same way as f_x - we simply consider all other variables as constants and differentiate with respect to the relevant one.

Example 3.1. Find a formula for f_{x_i} for arbitrary i given that $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$.

Notice that this equation is linear in all directions with a slope of 1, so for any i we have $f_{x_i} = 1$.

4. Higher Derivatives

Just as with single variable, we can differentiate a function many times. However, with multivariable functions, there are lots of different variables we can differentiate with respect to, so we need to be careful to give suitable notation so we know when calculating higher derivatives which variable to differentiate with respect to and when.

Result 4.1. Suppose f is a function of 2 variables x and y. We define the second partial derivatives of f as follows:

(i) The second partial derivative with respect to x:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial x}$$

(ii) The mixed partial derivative with respect to y and then x :

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

(iii) The mixed partial derivative with respect to x and then y :

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

We define the second partial derivatives with respect to y in a similar way, and in general we can define the kth partial derivative of a function $f(x_1, \ldots, x_n)$ with respect to different variables in a similar way. Calculation is straight forward.

Example 4.2. Calculate f_{xx}, f_{yy}, f_{xy} and f_{yx} for the following:

(i) $f(x, y) = x^2y^2 + xy$

$$f_{xx} = 2y^2, f_{yy} = 2x^2, f_{xy} = 4xy + 1 \text{ and } f_{yx} = 4xy + 1$$

(ii) $f(x, y) = \frac{y}{x} + y^2$

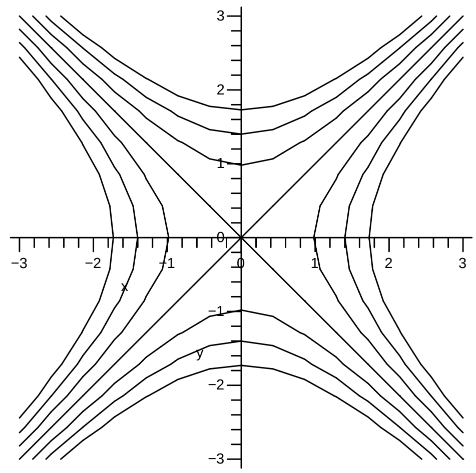
$$f_{xx} = \frac{2y}{x^3}, f_{yy} = 2, f_{xy} = -\frac{1}{x^2} \text{ and } f_{yx} = -\frac{1}{x^2}$$

Notice that in both these cases, we have $f_{xy} = f_{yx}$. Though this is not always true, it is true provided we add certain additional conditions. Specifically, we have the following result:

Result 4.3. (Clairauts Theorem) If the functions f_{xy} and f_{yx} are continuous, then $f_{yx} = f_{xy}$.

We finish with an example of how to determine partial derivatives when looking at graphs.

Example 4.4. You are stood at the point $(1, 1)$ on top of a hill. What are the signs of the following partial derivatives?



f_x - negative - when you walk in the x-direction, you will move down
f_y- negative - when you walk in the y-direction, you will move down
f_{xx} - negative - contours are getting closer together, so the hill is getting steeper, so the graph is concave down
f_{yy}- negative - contours are getting closer together, so the hill is getting steeper, so the graph is concave down
f_{xy} - positive - f_{xy} is te rate of change of f_x as y increase. If we fix and and move in the positive y-direction, then the contours in the x-direction are getting closer together. Notice also that the values of f_x are positive, so the positive values of f_x are getting larger, as y increases, so f_{xy} is positive.