

Section 15.4

Tangent Planes and Linear Approximations

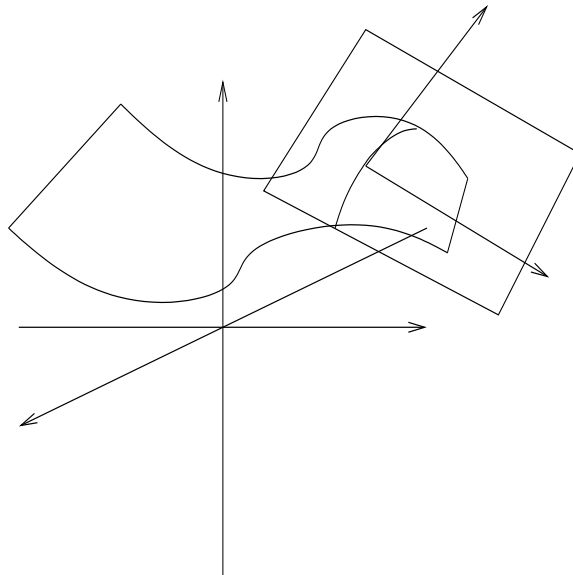
“Generalizing the Tangent Line”

Recall that one of the primary results in Calculus 1 was to approximate functions with lines. Specifically, the tangent line to a function at a point. We generalize this idea to functions of more than one variable.

1. TANGENT PLANES

Suppose a surface S has equation $z = f(x, y)$ where f is continuous and differentiable. Let $P(x_0, y_0, z_0)$ be a point on S . We want to define the tangent plane of S at P to be the plane which best approximates S at P . We do this as follows:

- In the plane $y = y_0$, there is a 2-d curve defined by $z = f(x, y_0)$. At the point P , we can find the tangent line to this curve using partial derivatives - call it T_1 .
- In the plane $x = x_0$, there is a 2-d curve defined by $z = f(x_0, y)$. At the point P , we can find the tangent line to this curve using partial derivatives - call it T_2 .



- For T_1 and T_2 , we can find the direction vectors. Specifically, we shall have $\vec{v}_1 = \vec{i} + f_x \vec{k}$ as the direction vector in the x direction and $\vec{v}_2 = \vec{j} + f_y \vec{k}$ in the y -direction (WHY?)
- We define the tangent plane T to be the plane which contains both vectors \vec{v}_1 and \vec{v}_2 .

Note that this will always give use a plane because the two vectors determined above will never be parallel and neither will ever be $\vec{0}$ (WHY?). Calculation of an equation for the tangent plane is fairly

straight forward using vector operations. Specifically, we first find the normal vector to the plane:

$$\begin{aligned}\vec{v}_1 \times \vec{v}_2 &= (\vec{i} + f_x \vec{k}) \times (\vec{j} + f_y \vec{k}) = \vec{i} \times \vec{j} + \vec{i} \times f_y \vec{k} + f_x \vec{k} \times \vec{j} + f_x \vec{k} \times f_y \vec{k} \\ &= -f_x \vec{i} - f_y \vec{j} + \vec{k}\end{aligned}$$

Then we apply the "point-slope" form for planes giving the following equation:

$$-f_x(x - x_0) - f_y(y - y_0) + (z - z_0) = 0$$

However, since we are considering this as a tangent plane to a surface at a point, we usually write it in a different way to reflect the tangent line equation we usually use in single variable calculus. Specifically, we have:

Result 1.1. Suppose f has continuous partial derivatives. An equation for the tangent plane to $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0).$$

Example 1.2. Find an equation for the tangent plane to $f(x, y) = x^2 - y^2$ at the point $(1, 1, 0)$.

We have $f_x(1, 1) = 2$, and $f_y(1, 1) = -2$, so the tangent plane has equation $2(x - 1) - 2(y - 1) = z$.

Example 1.3. You are trying to approximate the tangent plane to a hill at the point you are stood. You know if you move north 2 miles, you will gain 1/2 mile elevation and if you walk east 3 miles, you lose 1/2 mile elevation. Assuming you are stood at the origin, what is an approximation for the equation of the tangent plane to the hill where you are stood.

Let x be in the East direction and y in the North direction. Then $f_x \sim (1/2)/3 = 1/6$ and $f_y = (1/2)/2 = 1/4$. Thus the equation for the tangent plane at this point will be

$$\frac{1}{6}x + \frac{1}{4}y = z.$$

2. LINEAR APPROXIMATIONS AND DIFFERENTIABILITY

In the previous sections, we derived the equation for the equation of a tangent plane to a surface at a point. One of the most important uses for tangent planes is that they approximate values of functions provided you are fairly close to the point of the tangent plane. This motivates the following definition.

Definition 2.1. Suppose $f(x, y)$ has continuous partial derivatives and $P(x_0, y_0, z_0)$ is a point on S . Then the tangent plane

$$L(x, y) = f_x(x - x_0) + f_y(y - y_0) + z_0$$

is called the linearization or linear approximation of f at (x_0, y_0) . Specifically, provided (a, b) is close to (x_0, y_0) , we have $L(a, b) \sim f(a, b)$.

Recall that a function of a single variable is differentiable if the derivative exists, or equivalently, the tangent line exists. This is not quite the same for functions of more than one variable because there is more than one variable to differentiate with respect to. Therefore, we need to modify the definition of the derivative. Though the original definition does not work, the extended one does - specifically, we have the following:

Definition 2.2. We say f is differentiable at (a, b) if the tangent plane exists at (a, b) .

Sufficient conditions for a function $f(x, y)$ to be differentiable are the following:

Result 2.3. If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

Example 2.4. Show that $f(x, y) = e^xy$ is differentiable, write down the linear approximation to f at $(0, 3)$ and use it to approximate $f(0.1, 3.1)$.

We have $f_x = e^xy$ and $f_y = e^x$, both of which are continuous. Thus by the last result, f is differentiable.

We have $f_x(0, 3) = 3$ and $f_y(0, 3) = 1$, so the linear approximation is $L(x, y) = 3x + (y - 3) + e^0 * 3 = 3x + y$. This gives $L(0.1, 3.1) = 0.3 + 3.1 = 3.4$ (note that $3.1 * e^{0.1} = 3.42$, so there is an error of 0.02).

3. DIFFERENTIALS

Recall that in single variable calculus, the differential of a function is the function $dy = f'(x)dx$, and it is used to measure small changes in a function y given a small change in the variable x (so dy and dx are considered variables). As with single variables, we can define the differential of a function of two variables (or more). We define it in a similar way to the single variable case.

Definition 3.1. The differential of f at a point is defined to be

$$dz = f_x dx + f_y dy$$

where dx and dy are variables.

We think of dx and dy as changes in x and y , and dz as measuring the resultant change in z due to these changes in x and y . One important application of differentials is the ability they have to measure error terms when we are given approximate values of a function and bounds on the error. We illustrate.

Example 3.2. Find the differential of $f(x, y) = e^{xy} + y^2x$.

$$dz = f_x dx + f_y dy = (ye^{xy} + y^2)dx + (xe^{xy} + 2xy)dy.$$

Example 3.3. The measurements of a closed rectangular box are measured as $H = 80\text{cm}$, $W = 60\text{cm}$ and $D = 50\text{cm}$ respectively with an error measurement of at most 0.1cm in each. Use differentials to estimate the maximum error in calculating the surface area of the box.

We have $SA = 2WH + 2WD + 2DH$ where W is the width, H is the height and D is the depth. This gives the differential $d(SA) = (2H + 2D)dW + (2W + 2D)dH + (2W + 2H)dD$. We know $H \sim 80$, $W \sim 60$ and $D \sim 50$, and $dH = dW = dD \leq 0.1$, so $d(SA) \leq .4 * 80 + .4 * 60 + .4 * 50 = 76$.

4. FUNCTIONS OF MORE VARIABLES

We can define the linear approximation of a function of n variables in exactly the same way as for two variables. Specifically, we define:

Definition 4.1. If the partial derivatives of $f(x_1, \dots, x_n)$ are continuous at $P(a_1, \dots, a_n, a_{n+1})$, we define the linear approximation of f at P to be the function

$$L(x_1, \dots, x_n) = f_{x_1}(x_1 - a_1) + \dots + f_{x_n}(x_n - a_n) + a_{n+1}$$

Calculations are identical to the case where f is a function of two variables.