Section 15.5
The Chain Rule

"How to differentiate compositions of functions"

Recall that in single variable calculus, if $f(g(x))$ is a composition of functions, then $\frac{df(g(x))}{dx} = f'(g(x))g'(x)$. For functions of more than one variable, this is no longer directly the case since there are many different variables we could differentiate with respect to. We need to generalize the idea of the chain rule to functions of more than one variable.

1. Tree Diagram

In order to develop the chain rule, we need a new idea called a tree diagram. We set it up as follows:

- Suppose f is a function of n variables, call them $x_1, \ldots, x_n$. Suppose in addition, the variables $x_1, \ldots, x_n$ are also functions of m other variables, call them $u_1, \ldots, u_m$ (and we could continue letting $u_1, \ldots, u_m$ depend upon other variables etc). Then we can construct a diagram which reflects these dependencies.
- On the top row of the diagram, we write f
- On the the next row we write the variables f directly depends upon $(x_1, \ldots, x_n)$.
- On the the next row the variables $x_1, \ldots, x_n$ depend upon $(u_1, \ldots, u_m)$.
- We continue to do this until all variables are written down. We draw branches between all entries in each row representing a dependency.
- On a branch connecting two variables, write down the derivative of the top variable by the bottom variable - be careful to specify whether it is a partial derivative or a real deriavive so for example, on the branch from x_1 to u_3, we would write $\frac{\partial x_1}{\partial u_3}$).

We illustrate with a couple of examples.

Example 1.1. (i) Draw the tree diagram for $f(x, y, z)$ if x, y, and z depend upon a single variable t.

$$\begin{tikzcd}
 & f \arrow[dl, "\frac{\partial f}{\partial x}"'] \arrow[d, "\frac{\partial f}{\partial y}"] \arrow[dr, "\frac{\partial f}{\partial z}"] & \\
x \arrow[dr, "\frac{dx}{dt}"'] & y \arrow[d, "\frac{dy}{dt}"] & z \arrow[dl, "\frac{dz}{dt}"] \\
 & t &
\end{tikzcd}$$

(*ii*) Draw a tree diagram for $g(x(u,v), y(u,v))$.

$$\begin{tikzcd}
 & f \arrow[dl, "\frac{\partial f}{\partial x}"'] \arrow[dr, "\frac{\partial f}{\partial y}"] & \\
x \arrow[d, "\frac{\partial x}{\partial u}"] \arrow[drr, "\frac{\partial x}{\partial v}"] & & y \arrow[d, "\frac{\partial y}{\partial v}"] \arrow[dll, "\frac{\partial y}{\partial u}"] \\
u & & v
\end{tikzcd}$$

The construction of these diagrams is fairly straight forward. The reason we use them is because it makes the chain rule extremely easy.

Result 1.2. Suppose f is a function which is a composition of other functions. Then if we want to differentiate f with respect to one of the variables we do the following:

- Locate all branches from f to the variable you want to differentiate with respect to. Take the product of all derivatives along each branch.
- Add up the products from above - this is the derivative of f with respect to that variable.

It seems complicated, but it is actually straight forward. We illustrate.

Example 1.3. (*i*) Find the derivative $\frac{df}{dt}$ for $f(x(t), y(t), z(t))$.

We already found the tree diagram for this. Taking the product and summing, we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

(*ii*) Find $\frac{\partial g}{\partial u}$ for $g(x(u,v), y(u,v))$.

Again, we have already found the tree diagram, so taking products and sums, we get

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}.$$

We finish by considering some explicit examples and applications.

Example 1.4. (*i*) Suppose $f(x,y) = x^2 - y^2$, $x(u,v) = \cos(u) - \sin(v)$ and $y(u,v) = u^2 v$. Find $\frac{\partial f}{\partial v}$.

Using a tree diagram, we get

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We can calculate each of these derivatives gicing

$$\frac{\partial f}{\partial u} = -2x\sin(u) - 2yuv.$$

(*ii*) The radius of a right circular cone is increasing at a rate of 1.8cm/s while its height is increasing at a rate of 2.5cm/s. At what rate is the volume of the cone increasing when the radius is 120cm and the height is 140cm?

We know $V = \frac{1}{3}\pi r^2 h$. We also know that r and h are functions of time t. We are looking for dV/dt, so using the chain rule, we get

$$\frac{dV}{dt} = \frac{\partial f}{\partial r}\frac{dh}{dt} + \frac{\partial f}{\partial h}\frac{dh}{dt}.$$

Calculating, we have $\frac{\partial f}{\partial r} = \frac{2}{3}\pi r h$ and $\frac{\partial f}{\partial h} = \frac{1}{3}\pi r^2$, so when $r = 120$ and $h = 140$, we have $\frac{\partial f}{\partial r} = 11200$ and $\frac{\partial f}{\partial h} = 4800$. Thus,

$$\frac{dV}{dt} = \frac{\partial f}{\partial r}\frac{dh}{dt} + \frac{\partial f}{\partial h}\frac{dh}{dt} = 11200 * 1.8\pi + 4800 * 2.5\pi = 32160\pi.$$

2. Application: Implicit Differentiation in Single Variable Calculus

We can use the chain rule in multivariable calculus to derive a formula for differentiation obtained by implicitly differentiating a function in single variable calculus. Specifically, if we are given any equation in x and y, we can always rewrite it as $F(x, y) = 0$ where F is a function of x and y (by simply moving the variables over one side of the equation). Differentiating both sides with respect to x and applying the chain rule, we get

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0 \text{ so } \frac{\partial dy}{\partial dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

We illustrate with an example.

Example 2.1. Find $\frac{dy}{dx}$ if $\sqrt{x} = y^2 x - 2$.

First, we write $F(x, y) = y^2 x - 2 - \sqrt{x}$. Then we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{-\frac{1}{2\sqrt{x}} + y^2}{2xy} = \frac{\frac{1}{2\sqrt{x}} - y^2}{2xy}.$$