Section 15.7
Maximum and Minimum Values

"Finding minimum and maximum values for functions of more than one variable"

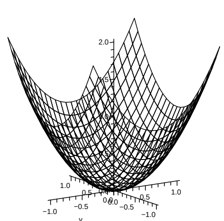
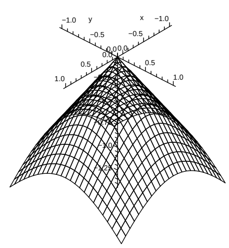
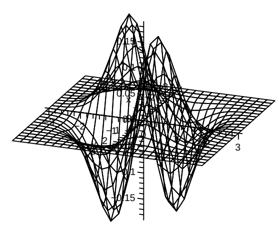
One important application in single variable calculus is finding the minimum and maximum values of a function. This can be done through the use of the first and second derivative tests. In this section we shall develop tests to determine the minimum and maximum values of a function of two variables.

1. Minimum and Maximum Values

First we need a formal definition of a minimum and maximum value.

Definition 1.1. A function of two variables $f(x, y)$ has a local maximum value at (a, b) if $f(x, y) \leqslant f(a, b)$ whenever (x, y) is near (a, b). It is said to have a local minimum value at (a, b) if $f(x, y) \geqslant f(a, b)$ whenever (x, y) is near (a, b). If the inequalities hold for all points (x, y) in the domain, we call (a, b) an absolute maximum or minimum of $f(x, y)$. We call all such points extreme values.

The following are examples of minimum and maximums.

The following result will help us to find minimum and maximum values for $f(x, y)$.

Result 1.2. If f has a local maximum or minimum at (a, b) and the first order partial derivatives exist, then $f_y(a, b) = f_x(a, b) = 0$.

This is a direct generalization of the single variable case. We make a couple of important observations:

(i) Just because the partial derivatives are 0 does not mean there is a min or max (we shall look at some examples of this later).
(ii) If the partial derivatives do not exist does not mean there is not a min or max (see the cone example above).
(iii) Geometrically, if there is a min or max at (a, b) and the partial derivatives exist, then the tangent plane will be horizontal.

(*iv*) Since mins and maxes occur when the partial derivatives are 0, we give the points where this occurs a special name. Specifically, if $f_x(a,b) = f_y(a,b) = 0$, we call (a,b) a critical point or stationary point of f.

(*v*) Using the result, mins and maxes occur either when both partial derivatives are 0 or when one is undefined. Therefore, we need to get good at finding such points.

Example 1.3. Find the extreme values of $f(x,y) = 9-2x+4y-x^2-4y^2$

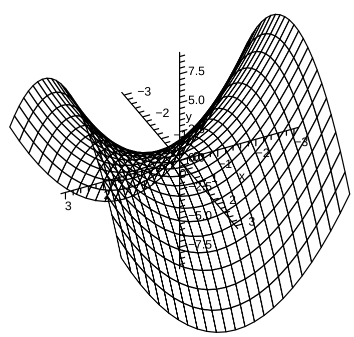
We have $f_x = -2-2x = 0$ when $x = -1$, and $f_y = 4-8y = 0$ when $y = 1/2$. Thus there is a single critical value for $f(x,y)$ at the point $(-1, 1/2)$. Simplifying, we have

$$f(x,y) = 11 - (x+1)^2 - 4(y-1/2)^2$$

which is an upside down parabolic bowl. Thus the critical value is a maximum with value 11.

Example 1.4. Find the extreme values of $f(x,y) = x^2 - y^2$

We have $f_x = 2x = 0$ when $x = 0$, and $f_y = -2y = 0$ when $y = 0$. Thus there is a single critical value for $f(x,y)$ at the point $(0,0)$. Looking at the graph, this is neither a minimum or a maximum - this is a special type of graph known as a saddle where the function is increasing in one direction and decreasing in the other.



In order to determine whether a function has a min or max, we use the second derivative test for functions of more than one variable.

Result 1.5. Suppose the second partial derivatives of f are continuous on a disc with center (a,b) and suppose $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(*i*) if $D > 0$ and $f_{xx} > 0$, then $f(a,b)$ is a local minimum

(*ii*) if $D > 0$ and $f_{xx} < 0$, then $f(a,b)$ is a local maximum

(*iii*) if $D < 0$, then $f(a,b)$ is neither a maximum or a minimum
(*iv*) if $D = 0$, the test is inconclusive.

We consider some examples.

Example 1.6. Find the saddle points and local mins and maxes of $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$.

We have $f_x = 6x^2 + y^2 + 10x$ and $f_y = 2xy + 2y = 2y(x+1)$. This means that $f_y = 0$ when either $x = -1$ or $y = 0$. If $y = 0$, we have $f_x = 6x^2 + 10x = 6x(x + 5/3)$ giving $x = 0$ or $x = -5/3$, so $(0,0)$ and $(-5/3, 0)$ are critical points. If $x = -1$, we have $f_x = 6 + y^2 - 10 = 0$ or $y = \pm 2$ giving $(-1, \pm 2)$ as critical points. We consider each point individually.

Next, we have $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$ and $f_{xy} = 2y$, so $D(x,y) = (12x+10)(2x+2) - 4y^2$. To finish, we test each point:

(*i*) $(0,0)$ - $D(0,0) = 20 > 0$ and $f_{xx}(0,0) = 10 > 0$, so $f(0,0)$ is a local min.
(*ii*) $(-5/3, 0)$ - $D(-5/3, 0) > 0$ since $12x+10$ and $2x+2$ will both be negative. Also, $f_{xx}(-5/3, 0) < 0$, so $f(-5/3, 0)$ is a local max.
(*iii*) $(-1, 2)$ - $D(-1,2) < 0$ so a saddle point
(*iv*) $(-1,-2)$ - $D(-1,-2) < 0$ so a saddle point

2. Absolute Maximum and Minimum Values

We may be interested in finding the largest max and smallest min of a function of more than one variable. If the domain is infinite, this is not necessarily possible. However, if the domain is bounded, this is.

Result 2.1. If f is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute max and min value in D.

This means that an absolute max or min takes place at either a boundary point or a critical point. Therefore to find the absolute max and min, we do the following:

(*i*) Find critical points in D.
(*ii*) Find the boundary points on D.
(*iii*) The largest of these is the max, the smallest is the min.

We illustrate with an example.

Example 2.2. Find three positive numbers whose sum is 100 and whose product is maximum.

We know $x + y + z = 100$ where $0 < x, y, z \leqslant 100$. Solving for z, we have $z = 100 - x - y$. The product of these numbers is xyz, which after substituting in, is a function of x and y, $P(x,y) = xy(100 - x - y) = 100xy - x^2y - xy^2$. Taking partials, we have

$$f_x = 100y - 2xy - y^2 = y(100 - 2x - y)$$

and

$$f_y = x(100 - 2y - x).$$

This gives either x or y equal to 0 (with the other variable allowed to vary), or we must have $100 - 2y - x = 0$ and $100 - 2x - y = 0$. These are linear equations, so solving, we get $x = 100/3$ and $y = 100/3$.

We shall check each of these points. We have $f_{xx} = -2y$, $f_{yy} = -2x$ and $f_{xy} = 100 - 2x - 2y$. Plugging in $x = 100/3$ and $y = 100/3$, we have $f_{xx} < 0$ and $D > 0$, so it is a max. Next, when either $x = 0$ or $y = 0$, $f = 0$ so is clearly minimal. To finish, we need to consider the boundary. However, since we are considering the problem for $0 \leqslant x, y \leqslant 100$, the boundary occurs when $x = 0$ and y is allowed to vary, or when $y = 0$ and x is allowed to vary. Such values are minimum values as shown above, and thus the absolute maximum must be $x = 100/3$ and $y = 100/3$.

Notice that in this last example, the problem of determining the behavior at the boundary was fairly straight forward since the function was constant on the boundary. More generally however, this is a much more difficult problem, so the final topic we shall consider in this chapter is how to solve this problem more generally.