Section 15.8
Lagrange Multipliers

"Finding minimum and maximum values for functions subject to certain constraints"

Recall that in the last section, to determine the absolute maximum and minimum values of a function, we needed to consider the values of that function on the boundary. Such a problem is equivalent to determining the minimum and maximum values of a function subject to some particular constraint. In this section, we shall consider this problem in more detail. We outline through an example. Consider the following situation which is typical in a business situation:

"A firm manufactures a commodity at two different factories. The total cost of manufacturing depends on the quantities q_1 and q_2 supplied by each factory expressed by the function

$$C(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500.$$

The companies objective is to produce 200 units while minimizing production costs. How many units should be supplied by each factory?"

Mathematically, we are asking to minimize a function (namely the cost function $C(q_1, q_2)$), given certain restrictions on the inputs - namely that we want precisely 200 units, or rather $q_1 + q_2 = 200$. Therefore, in mathematical terms, we interpret the problems as follows:

"Minimize the function

$$C(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500$$

subject to the constraint $q_1 + q_2 = 200$."

This is not a standard minimization problem because we have very specific restrictions on the domain. This means we cannot just apply the derivative test and instead need to develop a new way to solve such problems.

1. The Method of Lagrange Multipliers

In order to solve a minimization or maximization problem of a function f subject to constraints on its domain, we use the following method (we present the result for a function of three variables, but it works equally well for a function of two variables):

Result 1.1. To find the minimum and maximum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$), we do the following:

(i) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, , y, z) = k.$$

(We call λ a Lagrange multiplier, f the objective function and $g = k$ the constraint).

(ii) Evaluate f at all of the points found in step (i). The largest of these is the maximum and the smallest is the minimum of f subject to these constraints.

We illustrate with our business example.

Example 1.2. Minimize the function $C(q_1, q_2) = 2q_1^2 + q_1q_2 + q_2^2 + 500$ subject to the constraint $q_1 + q_2 = 200$.

Here we have $g(q_1, q_2) = q_1 + q_2$ and we have the constraint $g(q_1, q_2) = 200$. Calculating the necessary formulas, we have $\nabla C = (4q_1 + q_2)\vec{i} + (2q_2 + q_1)\vec{j}$ and $\nabla g = \vec{i} + \vec{j}$, so we need to solve the solutions to the two equations

$$(4q_1 + q_2)\vec{i} + (2q_2 + q_1)\vec{j} = \lambda\vec{i} + \lambda\vec{j}$$

and

$$q_1 + q_2 = 200.$$

This means we need to solve the three simultaneous equations

$$\begin{aligned} 4q_1 + q_2 &= \lambda \\ 2q_2 + q_1 &= \lambda \\ q_1 + q_2 &= 200 \end{aligned}$$

Using simple elimination on the first two, we get

$$3q_1 - q_2 = 0 \text{ or } q_2 = 3q_1.$$

Using the fact that $q_1 + q_2 = 200$, it follows that $4q_1 = 200$, so $q_1 = 50$ and $q_2 = 150$. Evaluating, we get $C(50, 150) = 35500$.

In principle, this could be a maximum value. However, to see it is a min, we note that $f(0, 200) = 40,500 > 35500$.

The method of Lagrange multipliers, though long, is not very technical - it mainly involves solving equations. We look at a more involved question.

Example 1.3. Find the maximum volumes of a rectangular box whose surface area is $600cm^2$.

If x, y, z denote the side lengths of the box, then we are trying to minimize and maximize the function $V(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2xy + 2xz + 2yz = N$. We apply the method of Lagrange multipliers:

$$\nabla V = yz\vec{i} + xz\vec{j} + xy\vec{k}, \text{ and } \nabla g = 2(y + z)\vec{i} + 2(x + z)\vec{j} + 2(x + y)\vec{k}$$

so we are solving the equations

$$\begin{aligned} yz &= 2(y+z)\lambda \\ xz &= 2(x+z)\lambda \\ xy &= 2(x+y)\lambda \\ 2xy + 2xz + 2yz &= 600 \end{aligned}$$

Solving the first two equations for y and x respectively, we get $y = \lambda/(2\lambda+z)$ and $x = \lambda/(2\lambda+z)$, so it follows that $x = y$. Likewise, we can show that $x = z$, so we must have $x = y = z$. Putting this in the last equation, we have $6x^2 = 600$, or $x = 10$. Thus there is a critical point at $(10, 10, 10)$ (when the box is a cube) and $V(10, 10, 10) = 10^3 = 1000$. To see this is a maximum, observe that if we choose either x, y, or $z = 0$, then the volume will be zero.

Lagrange multipliers can also be used to determine mins and maxes when there is more than one constraint on the function. The method is almost identical, the only difference being that if we are trying to maximize or minimize $f(x, y, z)$ subject to $g(x, y, z) = k$ and $h(x, y, z) = c$, then we need to solve the following system of equations:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$
$$g(x, y, z) = k$$

and

$$h(x, y, z) = c.$$

We illustrate with an example.

Example 1.4. Find the mins and maxes of $f(x, y, z) = yz + xy$ subject to the two constraints $xy = 1$ and $y^2 + z^2 = 1$.

We use Lagrange multipliers: $\nabla f = y\vec{i} + (x+z)\vec{j} + y\vec{k}$, $\nabla g = y\vec{i} + x\vec{j}$ and $\nabla h = 2y\vec{j} + 2z\vec{j}$. Thuse we are trying to solve the equations

$$y = \lambda y$$
$$(x+z) = \lambda x + 2y\mu$$
$$y = 2z\mu$$
$$xy = 1$$

and

$$y^2 + z^2 = 1.$$

Since $xy = 1$, we cannot have $y = 0$, so it follows that $\lambda = 1$. After This gives the following equations:

$$z = 2y\mu$$
$$y = 2z\mu$$
$$xy = 1$$

and

$$y^2 + z^2 = 1.$$

Using the first two equations, we have $z = 2y\mu = 2(2z\mu)\mu = 4z\mu^2$, so $\mu = \pm 1/2$. This also means that $z = \pm y$. Using the last equation, it follows that $z = \pm 1/\sqrt{2}$ and so $y = \pm 1/\sqrt{2}$. Using the last equation, this means $x = \pm\sqrt{2}$ dependent upon y. Thus the critical points are $(\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$, $(-\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2})$, $(\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2})$, and $(-\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2})$. Plugging into $f(x, y, z)$, we have

$$f(\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}) = \frac{3}{2}$$

$$f(-\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2}) = \frac{1}{2}$$

$$f(\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{2}$$

$$f(-\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2}) = \frac{3}{2}.$$

Thus the minimum value of f is $1/2$ and the maximum is $3/2$.