Section 16.1
Double Integrals over Rectangles

"Integrating Functions of more than one Variable"

In single variable calculus, we developed the definite integral as a way to measure the area under a graph. We shall generalize this idea to multivariable calculus. We shall start by reviewing the single variable definite integral.

1. Review

Suppose $f(x)$ is a function which is continuous on the interval $[a, b]$. Then to define the definite integral of f over $[a, b]$, we do the following:

(i) Break up the interval into n equally sized pieces of size $\Delta x = (b-a)/n$.

(ii) In each interval, fix some x value in that interval (perhaps the midpoint, the left hand endpoint or the right hand endpoint). Observe that the value $f(x_i)\Delta x$ is a rectangle which approximates the area under f in the ith subinterval.

(iii) Define the Riemann sum over the interval $[a, b]$ as the sum of the areas of these rectangles:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

(iv) If we choose more and more subintervals, the value gets closer and closer to the actual area.

Thus we define:

Definition 1.1. The definite integral of f from a to b is defined to be

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

2. Volumes and Double Integrals

We shall generalize the ideas of the definite integral to functions of two variables. Suppose the region R is a rectangle in the xy-plane: $R = [a, b]\times[c, d] = \{(x, y) \in \mathbb{R}^2 | a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$. We temporarily assume that $f(x, y) > 0$ on this intevral. We want to define a sum which approximates the volume under $f(x, y)$. We do this as follows:

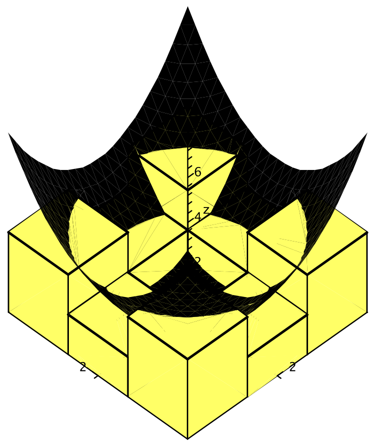
(i) Break up the interval $[a, b]$ into n subintervals of equal length $\Delta x = (b-a)/n$, and break up the interval $[c, d]$ into m subintervals of equal length $\Delta y = (d-c)/m$.

(ii) Suppose $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$ are two subintervals. Then the region

$$[x_i, x_{i+1}] \times [y_j, y_{j+1}] = \{(x, y) \in \mathbb{R}^2 | x_i \leqslant x \leqslant x_{i+1}, y_j \leqslant y \leqslant y_{j+1}\}$$

is a subrectangle of the rectangle R.

(iii) We can choose a sample point $[x_i, y_j]$ from this interval (perhaps the midpoint, or one of the corner points). Observe that if (x_i, y_j) is a sample point, then the product $f(x_i, y_j)\Delta x \Delta y$ approximates the volume under f above the subrectangle by a box (see illustration for $f(x, y) = x^2 + y^2$ with 9 subrectangles below and using the midpoint).

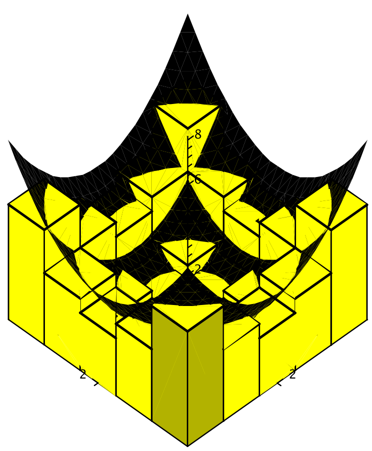


(iv) We define the Riemann sum over the interval R as the sum of the volumes of these boxes:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j)\Delta A$$

where $\Delta A = \Delta x \Delta y$.

(v) We observe that as we take smaller and smaller subintervals, the rectangles approximate the volume better and better (see illustration for $f(x, y) = x^2 + y^2$ with more subrectangles below and using the midpoint).



Our observations motivate the following definition.

Definition 2.1. The double integral of f over the rectangle R is

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\Delta A$$

provided this limit exists.

These observations suggest the following fact.

Result 2.2. If $f(x,y) \geqslant 0$ on the region R, then the volume under $f(x,y)$ above the region R is equal to

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\Delta A$$

provided this limit exists.

In general, the definite integral measures "weighted" volume where the volume below the xy-plane is counted negatively and the volume above the xy-plane is counted positively. We look an example to illustrate.

Example 2.3. Use a Riemann sum to estimate the volume under the function $f(x,y) = y^2 + 2x^2$ in the rectangle $[0,1] \times [1,2]$.

We break up the two intervals into subintervals of lengths 1/2 each giving a total of 4 subrectangles. Next, we need to choose reference points in each triangle - we shall choose the bottom left hand corner point in each rectangle, so $(0,1)$ in $[0,1/2]\times[1,1/2]$, $(0,3/2)$ in $[0,1/2]\times[3/2,2]$, $(1/2,1)$ in $[1/2,1]\times[1,3/2]$ and $(1/2,3/2)$ in $[1/2,1]\times[3/2,2]$. We tabluate the values of $f(x,y)$.

x \ y	1	3/2
0	1	9/4
1/2	5/4	5/2

Observe that $\Delta x = 1/2$ and $\Delta y = 1/2$, so $\Delta A = 1/4$. This means the volume can be approximated by the sum

$$(1 + 9/4 + 5/4 + 5/2)(1/4) = 7/4.$$

As with single variable, there are many different ways to estimate definite integrals like the midpoint rule, the trapeziod rule (which translates to the rhombus rule), simpsons rule (averaging the rhombus and midpoint rule) and so on. We illustrate with an example.

Example 2.4. Use the midpoint rule to estimate the definite integral of $f(x,y) = \ln(x^2 + y^2)$ over the rectangle $[-1,1] \times [2,4]$.

As with the last example, we break up the two intervals into subintervals of lengths 1 each giving a total of 4 subrectangles. Next, since we are using the midpoint rule, we need to choose the midpoints of each of our rectangles as our reference points - this gives $(-1/2, 5/2)$ in

$[-1,0]\times[2,3]$, $(-1/2,7/2)$ in $[-1,0]\times[3,4]$, $(1/2,5/2)$ in $[0,1]\times[3,4]$ and $(1/2,7/2)$ in $[0,1]\times[3,4]$. We tabluate the values of $f(x,y)$.

x \ y	5/2	7/2
-1/2	$\ln(13/2)$	$\ln(25/2)$
1/2	$\ln(13/2)$	$\ln(25/2)$

Observe that $\Delta x = 1$ and $\Delta y = 1$, so $\Delta A = 1$. This means the volume can be approximated by the sum

$$(\ln(13/2)+\ln(13/2)+\ln(25/2)+\ln(25/2))(1) = \ln(105625/16) \sim 8.79.$$

3. The Average Value of a Function

As with single variable, we can use definite integrals to determine the average value of a function over a region. In this case, the region of a function of two variables is a 2d region in the plane, so we have to adjust our definition of average value accordingly. Specifically, we define the average value as follows.

Definition 3.1. The average value of f over the rectangle R is

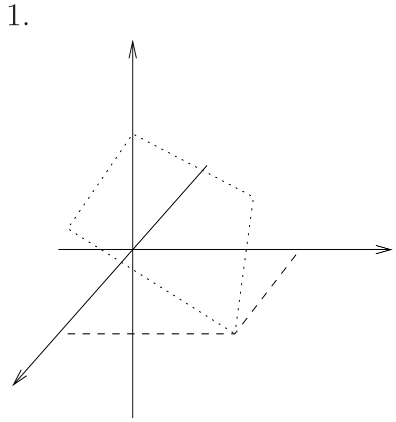
$$\frac{1}{A(R)}\int\int_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i,y_j)\Delta A$$

where $A(R)$ denotes the area of the rectangle R.

We finish with an example.

Example 3.2. Determine the exact volume under the graph of $f(x,y) = 2-x-y$ on the region $[0,1]\times[0,1]$ and use it to find the average value of $f(x,y)$ on this region.

Observe the function is a plane, which passes through the points $(1,1,0)$, $(0,0,2)$, $(0,1,1)$ and $(1,0,1)$ (see sketch). Notice that it cuts the box bounded by $[0,1]\times[0,1]$ from $0\leqslant z\leqslant 2$ exactly in half (by symmetry), so its volume is half the volume of this box. In particular, its volume will be $2*1*1/2 = 1$.



To calculate the average value, we divide the integral by the area of the rectangle R. In this case, the rectangle has area 1, so the average value of f is 1.