Section 16.2
Iterated Integrals

"The Fundamental Theorem of Calculus for functions of two variables"

Using the sum definition of the definite integral makes it very difficult to evaluate the definite integral. Recall however that in single variable, we stopped using the sum definition and instead applied the fundamental theorem of calculus to evaluate definite integrals. Specifically, we used the following theorem:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

In this section we want to generalize FTC so we can integrate functions of two variables over rectangles.

1. Evaluating Integrals Iteratively

An iterated integral is simply the process of integrating a function a number of different times with respect to different variables. Specifically, it can be defined as follows:

(i) Suppose that $f(x, y)$ is a function which is continuous on the rectangle $[a, b] \times [c, d]$.

(ii) If we integrate the function $f(x, y)$ with respect to just x from $x = a$ to $x = b$, the result is a function purely in terms of y:

$$A(y) = \int_a^b f(x, y)dx$$

We call the integral a partial integral of $f(x, y)$ with respect to x.

(iii) Since $A(y) = \int_a^b f(x, y)dx$ is a function of y, we can integrate it with respect to y from c to d which results in some number. We denote it by

$$\int_c^d \int_a^b f(x, y)dxdy$$

and call it an iterated integral.

(iv) We could also have integrated with respect to y first and then x to obtain another iterated integral.

The importance of iterated integrals is the following result called Fubini's Theorem.

Result 1.1. If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\int\int_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

A formal proof of Fubini's theorem is very difficult, but we can use the definition of the definite integral to determine an idea of why it is true. We know

$$\int\int_R f(x,y)dA = \lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_j)\Delta A =$$

$$\lim_{n,m\to\infty}\sum_{i=1}^{n}\left[\sum_{j=1}^{m} f(x_i,y_j)\Delta y\right]\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\lim_{m\to\infty}\sum_{j=1}^{m} f(x_i,y_j)\Delta y\right]\Delta x$$

Observe that the limit of the sum

$$\lim_{m\to\infty}\sum_{j=1}^{m} f(x_i,y_j)\Delta y$$

is the single variable definition of the integral. In particular, it must be equal to

$$\int_c^d f(x_i,y)dy$$

where i varies over the second sum. Then we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\lim_{m\to\infty}\sum_{j=1}^{m} f(x_i,y_j)\Delta y\right]\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\int_c^d f(x_i,y)dy\right]\Delta x.$$

Now the integral

$$\int_c^d f(x_i,y)dy$$

is a function of x, so the sum of the limit

$$\lim_{m\to\infty}\sum_{i=1}^{n}\int_c^d f(x_i,y)\Delta x$$

is the single variable definition of the integral, so we get

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\int_c^d f(x_i,y)dy\right]\Delta x = \int_a^b\int_c^d f(x,y)dydx.$$

To show that we can integrate the other way around, we simply reorder the sum.

In a nutshell, Fubini's theorem tells us we can evaluate the double integral of a function over a rectangle by simply using an iterated integral and the order of integration is not a factor. The importance of this fact is that double integrals can be solved using simple single variable integration methods. We illustrate with a number of detailed examples.

Example 1.2. Calculate

$$\int_R x^2 + y^2 dA$$

where $R = [1,2]\times[-1,0]$.

We use iterated integrals:

$$\int_R x^2+y^2dA = \int_{-1}^{0}\int_{1}^{2} x^2+y^2dxdy = \int_{-1}^{0}\left(\frac{x^3}{3}+xy^2\Big|_1^2\right)dy$$

$$= \int_{-1}^{0}\frac{8}{3}+2y^2-\frac{1}{3}-y^2dy = \int_{-1}^{0}\frac{7}{3}+y^2dy = \frac{7y}{3}+\frac{y^3}{3}\Big|_{-1}^{0} = \frac{8}{3}$$

Example 1.3. Calculate

$$\int_R \frac{1+x^2}{1+y^2}dA$$

where $R=[0,1]\times[0,1]$.

We use iterated integrals:

$$\int_R \frac{1+x^2}{1+y^2} = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}dxdy = \int_0^1\left(\frac{x+x^3/3}{1+y^2}\Big|_0^1\right)dy$$

$$= \frac{4}{3}\int_0^1 \frac{1}{1+y^2}dy = \frac{4}{3}\arctan y\Big|_0^1 = \frac{\pi}{3}$$

Example 1.4. Evaluate the integral

$$\int_0^2\int_0^{\pi/2} x\sin(y)dydx$$

Calculating, we have

$$\int_0^2\int_0^{\pi/2} x\sin(y)dydx = \int_0^2 -x\cos(y)|_0^{\pi/2}dx = \int_0^2 xdx = \frac{x^2}{2}|_0^2 = 4$$

Example 1.5. Evaluate

$$\int\int_R 6x^2y^3-5y^4dA$$

where $R=[0,3]\times[0,1]$

In this case, an order has not been specified, so we shall integrate with respect to x and then y (Fubibi's Theorem guarantees the answer will be the same regardless):

$$\int\int_R 6x^2y^3-5y^4dA = \int_0^1\int_0^3 6x^2y^3-5y^4dxdy = \int_0^1 2x^3y^3-5xy^4|_0^3dy$$

$$= \int_0^1 54y^3-15y^4dy = \frac{27}{2}y^4-3y^5|_0^1 = \frac{21}{2}$$

Example 1.6. Find the volume of the solid bounded by the paraboloid $z = 1 + (x-1)^2 + 4y^2$ and the planes $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

Since the paraboloid only takes positive values, the volume of the region described will simply be the integral of $f(x, y) = 1+(x-1)^2+4y^2$ over the rectangle $[0, 1] \times [0, 1]$. We calculate:

$$\int_0^1 \int_0^1 1 + (x-1)^2 + 4y^2 dxdy = \int_0^1 \left(x + \frac{(x-1)^3}{3} + 4xy^2 \right)_0^1 dy$$

$$= \int_0^1 \left(1 + 4y^2 + \frac{1}{3}\right)_0^1 dy = \int_0^1 \left(\frac{4}{3} + 4y^2\right) dy = \left(\frac{4y}{3} + \frac{4y^3}{3}\right)_0^1 = \frac{8}{3}$$

Sometimes it may be useful to switch the direction of integration as the next example illustrates.

Example 1.7. Evaluate

$$\int\int_R \frac{x}{1+xy} dA$$

where $R = [0, 1] \times [0, 1]$

If we integrate this function with respect to x first, we shall have to do polynomial division first. Therefore, we shall integrate with respect to y first. We have

$$\int_0^1 \int_0^1 \frac{x}{1+xy} dydx = \int_0^1 \ln(1+xy)|_0^1 dy = \int_0^1 \ln(1+x) dy$$

$$= ((x+1)\ln(x+1) - x)_0^1 = 2\ln(2) - 1.$$

Recall that to integrate $\ln(x+1)$, we need to use integration by parts. Specifically,

$$u = \ln(x+1) \quad dv = 1$$
$$du = \tfrac{1}{x+1} \quad v = x$$

so we get

$$\int \ln(x+1)dx = x\ln(x+1) - \int \frac{x}{x+1} dx = x\ln(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x\ln(x+1) - (x - \ln(x+1)) = (x+1)\ln(x+1) - x$$