Section 16.3
Double Integrals over General Regions

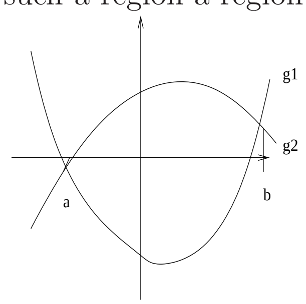
"Not every region is a rectangle"

In the last two sections we considered the problem of integrating a function of two variables over a rectangle. This situation however is fairly unrealistic since most regions in 2-space are not rectangles. We need to develop a way to calculate integrals over general regions in the plane. In order to integrate a function $f(x, y)$ over an arbitrary region R, we shall develop two related methods dependent upon the structure of the region R.

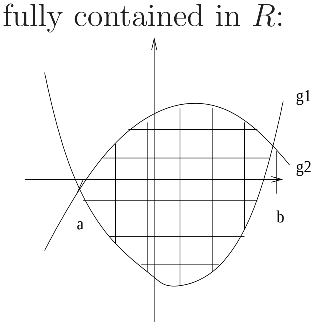
1. Integrals of types 1 and 2

The first type of region we consider is when the boundary of the region consists of two functions of x. We describe below.

(*i*) Suppose we want to integrate a function $f(x, y)$ over some arbitrary region R in the plane which is bounded between two functions $g_1(x)$ and $g_2(x)$ and between a and b as illustrated below. We call such a region a region of **type 1**.



(*ii*) We shall set up a Riemann sum over this region to evaluate the definite integral. First we subdivide the region up into small rectangles as illustrated where we do not count the rectangles which are not fully contained in R:



(*iii*) As usual, we take reference points in each subrectangle, and take a Riemann sum over all the rectangles multiplying the area of the rectangle by the function value at the reference point in the rectangle. Notice however, that this case is different to the case when we are integrating over rectangles - specifically, the number of rectangles and the values y is allowed to take depend upon x. Specifically, if we choose x_i as a reference point along a column of rectangles, then the y values are bounded between $g_1(x_i)$ and $g_2(x_i)$, so all rectangles must be within these bounds. This must be reflected in any Riemann sum we are using to try to define an integral. Therefore, we can define

$$\int\int_R f(x,y)dA \sim \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_j)\Delta y\Delta x$$

where the second sum is over the rectangles bounded by $g_1(x_i)$ and $g_2(x_i)$.

(iv) Taking smaller and smaller rectangles, the answer becomes more exact because the rectangles approximate the region better and better. Taking them infinitely small, we get the following iterated integrals:

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

We can use a similar argument if the region is bounded by functions in the y direction (regions of type 2), and thus we get the following important way of integrating functions of two variables over general regions:

Result 1.1. Suppose $f(x,y)$ is a continuous function on a region R.

(i) If the region is of **Type 1**, so bounded above and below by functions $g_1(x)$ and $g_2(x)$ and between $x=a$ and $x=b$, then

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

(ii) If the region is of **Type 2**, so bounded to the left and right by functions $h_1(y)$ and $h_2(y)$ and between $y=c$ and $y=d$, then

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

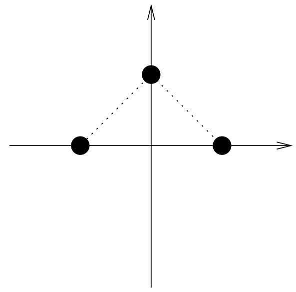
We illustrate with an example.

Example 1.2. Evaluate the integral

$$\int\int_R 2xdA$$

where R is the triangle with vertices $(-1,0)$, $(0,1)$ and $(1,0)$.

First we sketch the region.



Observe that this triangle is bounded between $0 \leqslant y \leqslant 1$ and the lines $y-1 \leqslant x \leqslant 1-y$. Therefore it is an integral of type 2, so

$$\int\int_R 2x dA = \int_0^1 \int_{y-1}^{1-y} 2x dx dy = \int_0^1 x^2 \Big|_{y-1}^{1-y} dy$$

$$= \int_0^1 [(1-y)^2 - (y-1)^2] dy = \int_0^1 [(y^2-2y+1)-(y^2-2y+1)] dy$$

$$= \int_0^1 0 dy = 0.$$

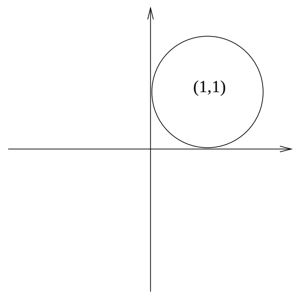
We could have also seen this from the symmetry across the x-axis!

As we observed in the construction of an iterated integral over an arbitrary region, the order of integration is extremely important depending upon whether it is a region of type 1 or type 2. Therefore, it is a good idea to have a few general checkpoints and useful facts about iterated integrals on arbitrary regions to avoid any silly errors. When setting up an iterated integral over a non-rectangular region, always check the following:

(*i*) The limits on the outer integral must be constants.
(*ii*) The limits on the inner integral should be functions of either x for type 1 or y for type 2.
(*iii*) Integrals are additive over regions - we can break a region up into pieces and add up the integral over each piece to get the integral over the whole region.
(*iv*) If a region is not type 1 or type 2, it can always be broken up into more integrals which are type 1 or 2 and then we can use the additivity of the integral to evaluate it.
(*v*) Much of the time, the difficulty is in setting up integrals and not actually evaluating them.

To illustrate the last point, we do a couple of examples.

Example 1.3. Set up an integral for the function $f(x, y)$ over the following regions.



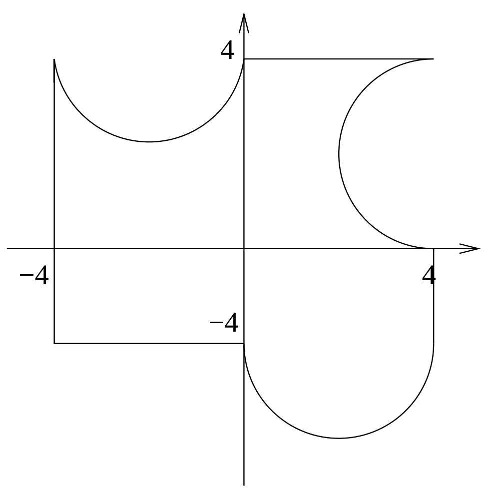
(*i*) This region is a circle of radius 1 centered at the point $(1, 1)$, so it has equation $(x-1)^2 + (y-1)^2 = 1$. It can be realized as either a type 2 or a type 1 integral. Realizing it as a type 2, we have $0 \leqslant y \leqslant 2$ and $1 - \sqrt{1-(y-1)^2} \leqslant x \leqslant 1 + \sqrt{1-(y-1)^2}$, so the integral will be

$$\int_0^2 \int_{1-\sqrt{1-(y-1)^2}}^{1+\sqrt{1-(y-1)^2}} f(x,y)dxdy.$$

Realizing it as a type 1 region, we have, $0 \leqslant x \leqslant 2$ and $1 - \sqrt{1-(x-1)^2} \leqslant y \leqslant 1 + \sqrt{1-(x-1)^2}$, so the integral will be

$$\int_0^2 \int_{1-\sqrt{1-(x-1)^2}}^{1+\sqrt{1-(x-1)^2}} f(x,y)dxdy.$$

(*ii*) Consider the following region which is neither type 1 or type 2.



This region needs to be broken up into different pieces. For $-4 \leqslant x \leqslant 0$, we can make it a type 1 integral. Specifically, we have $-4 \leqslant y \leqslant 4 - \sqrt{4-(x+2)^2}$. The fourth quadrant can also be made into a type 1 region with $0 \leqslant x \leqslant 4$ and $-4 - \sqrt{4-(x-2)^2} \leqslant y \leqslant 0$. The first quadrant can be made into a type 2 region with $0 \leqslant y \leqslant 4$ and $0 \leqslant x \leqslant$

$4-\sqrt{4-(y-2)^2}$. Putting these together, we have

$$\int\int_R f(x,y)dA = \int_{-4}^{0}\int_{-4}^{4-\sqrt{4-(x+2)^2}} f(x,y)dydx +$$

$$\int_0^4 \int_{-4-\sqrt{4-(x-2)^2}}^{0} f(x,y)dydx + \int_0^4 \int_0^{4-\sqrt{4-(y-2)^2}} f(x,y)dxdy$$

We finish with a couple more examples.

Example 1.4. Evaluate the integral

$$\int\int_R (3x+2y)dA$$

where R is quarter of the unit circle centered at the origin in the positive x and y-directions.

This region is either type 1 or type 2. Using type 1, we have $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant \sqrt{1-x^2}$, so

$$\int\int_R (3x+2y)dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (3x+2y)dydx$$

$$= \int_0^1 3xy + y^2\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 (3x\sqrt{1-x^2} + 1 - x^2)dx$$

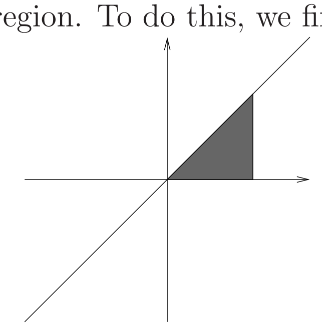
$$= -(1-x^2)^{\frac{3}{2}} + x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} - (-1) = \frac{5}{3}$$

Other examples to consider in groups:

(i) Evaluate

$$\int_0^1 \int_y^1 \sin x^2 dxdy$$

Note that we cannot integrate this function with respect to x - it is a function which does not have an algebraic antiderivative, so it looks like we cannot solve this problem. However, there is nothing to stop us from reversing the order of integration **provided** we can change the bounds as a type 2 region into a type 1 region. To do this, we first sketch the region:



As a type 1 region, this region will have the bounds $0 \leqslant y \leqslant x$ and $0 \leqslant x \leqslant 1$. Therefore, we have

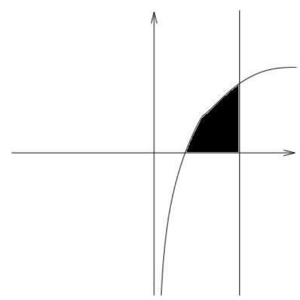
$$\int_0^1 \int_y^1 \sin x^2 dxdy = \int_0^1 \int_0^x \sin x^2 dydx = \int_0^1 \sin(x^2)y|_0^x dx$$

$$= \int_0^1 x\sin(x^2)ydx = -\frac{\cos(x^2)}{2}\bigg|_0^1 = -\frac{\cos(1)}{2} + \frac{\cos(0)}{2} = \frac{1}{2}(1-\cos(1))$$

(ii) Evaluate

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x} dxdy$$

The technique we used in the last example is very useful offering two possible ways to approach an integral. As with the last example, we cannot integrate this function with respect to x, so we shall use the same technique and transform this from a type 2 integral to a type 1 integral. First, we sketch the region:



As a type 1 region, this region will have the bounds $0 \leqslant y \leqslant \ln(x)$ and $1 \leqslant x \leqslant e$. Therefore, we have

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x} dxdy = \int_1^e \int_0^{\ln(x)} \frac{x}{\ln x} dydx = \int_1^e \frac{yx}{\ln(x)}\bigg|_0^{\ln(x)} dx$$

$$= \int_1^e \frac{\ln(x)x}{\ln(x)} dx = \int_1^e xdx = \frac{x^2}{2}\bigg|_1^e = \frac{1}{2}\left(e^2 - 1\right).$$

(iii) Evaluate

$$\int_1^4 \int_{\sqrt{y}}^y xy^3 dxdy$$

This integral can be evaluated directly:

$$\int_1^4 \int_{\sqrt{y}}^y xy^3 dxdy = \int_1^4 \frac{x^2}{2}y^3\bigg|_{\sqrt{y}}^y dy = \int_1^4 \frac{y^5}{2} - \frac{y^4}{2} dy$$

$$= \frac{y^6}{12} - \frac{y^5}{10}\bigg|_1^4 = \frac{1024}{3} - \frac{512}{5} - \frac{1}{12} + \frac{1}{10} = \frac{1775}{4}$$

(iv) Evaluate

$$\int_0^2 \int_0^x e^{x^2} dy dx$$

This integral can be evaluated directly:

$$\int_0^2 \int_0^x e^{x^2} dy dx = \int_0^2 y e^{x^2} \Big|_0^x dx = \int_0^2 x e^{x^2} dx$$

$$= \frac{e^{x^2}}{2} \Big|_0^2 = \frac{1}{2} \left(e^4 - 1 \right).$$