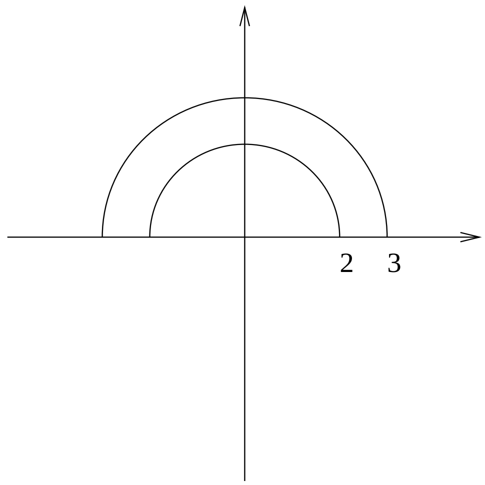
Section 16.4
Double Integrals in Polar Coordinates

"Integrating Functions over circular regions"

Suppose we want to integrate the function $f(x, y) = x^2$ over the following region:



We would have to break the region up into three pieces. On the first region we would have $-2 \leqslant x \leqslant 2$ and $\sqrt{4-x^2} \leqslant y \leqslant \sqrt{9-x^2}$, on the second region $-3 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant \sqrt{9-x^2}$ and on the last region $2 \leqslant x \leqslant 3$ and $0 \leqslant y \leqslant \sqrt{9-x^2}$. Thus we would have to evaluate the following integrals:

$$\int_{-3}^{-2} \int_{0}^{\sqrt{9-x^2}} x^2 dy dx + \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{9-x^2}} x^2 dy dx + \int_{2}^{3} \int_{0}^{\sqrt{9-x^2}} x^2 dy dx.$$

Not only was the process of setting up this integral laborious, but to evaluate the first integral,

$$\int_{-3}^{-2} \int_{0}^{\sqrt{9-x^2}} x^2 dy dx = \int_{-3}^{-2} x^2 \sqrt{9-x^2} dx$$

we would need to use a trigonometric substitution, which is itself a difficult and time consuming task. Therefore, we need to develop new techniques of integration to help us integrate functions over different types of regions.

Recall in Calculus 2 that when considering problems over circular regions, instead of using Cartesian coordinates, we can instead use polar coordinates to try to determine an integral. In this section we shall determine how to use polar coordinates to evaluate an integral.

1. Evaluating Integrals in Polar Coordinates over Polar Rectangles

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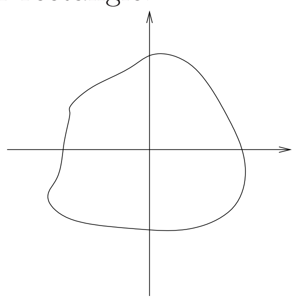
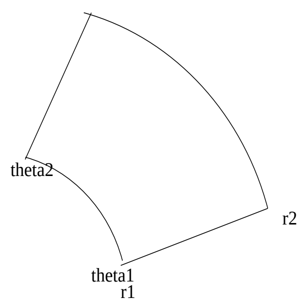
Suppose that $f(x, y)$ is continuous functions and we want to find the integral $\int\int_D f(x, y)dA$ over some region D as illustrated below. Then instead of integrating in Cartesian coordinates, we can use in polar coordinates. In order to determine exactly how, we shall first consider the problem over a polar rectangle.

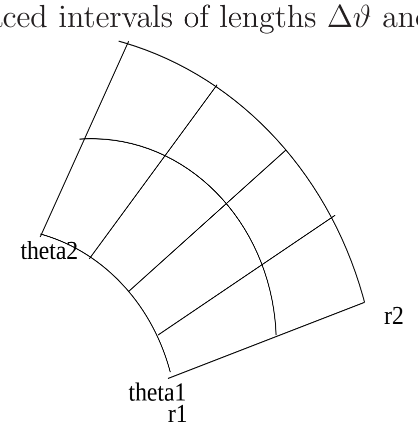


(i) Suppose that R is a polar rectangle (so ϑ is bounded between two constants and r is bounded between two constants) and $f(x, y)$ is a continuous function. Observe that R looks like the following:



(ii) First we convert the function $f(x, y)$ into a polar function by using the identities $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$.

(iii) Next we subdivide the polar rectangle into polar subrectangles by subdividing the intervals bounding ϑ and r into smaller equally spaced intervals of lengths $\Delta\vartheta$ and Δr (see below)



(*iv*) In each of the polar rectangles, choose a point (r_i, ϑ_j). Observe that the area of the polar rectangle can be approximated by the value $r_i \Delta\vartheta \Delta r$.

(*v*) We can take the double sum over the polar rectangle getting

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i, \vartheta_j) r \Delta r \Delta\vartheta$$

Notice that this is equal to the Riemann sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta A$$

over the same region.

(*vi*) Letting $m, n \to \infty$, these two sums are equal and so we get the following result:

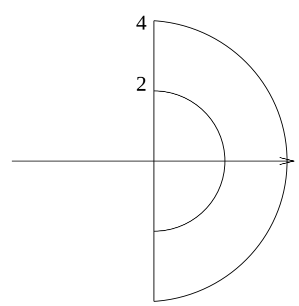
Result 1.1. If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$ and $0 \leqslant \alpha \leqslant \vartheta \leqslant \beta \leqslant 2\pi$, then

$$\int\int_R f(x,y) dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos(\vartheta), r\sin(\vartheta)) r dr d\vartheta$$

So we can use polar coordinate instead of cartesian coordinates (notice that we **must** include the r in the integrand). We illustrate with some examples.

Example 1.2. Set up integrals for $f(x, y)$ over the following regions:

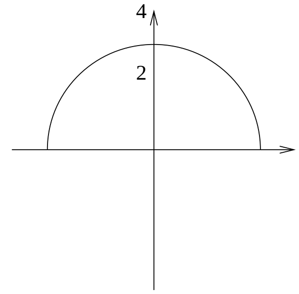
(*i*) Since $0 \leqslant \vartheta \leqslant 2\pi$, we need to break this integral up into two pieces.



We get

$$\int\int_R f(x,y) dA = \int_{0}^{\pi/2} \int_{2}^{4} f(r\cos(\vartheta), r\sin(\vartheta)) r dr d\vartheta + \int_{3\pi/2}^{2\pi} \int_{2}^{4} f(r\cos(\vartheta), r\sin(\vartheta)) r dr d\vartheta$$

(ii) This can be evaluated in one integral.



We get

$$\int\int_R f(x,y)dA = \int_0^{\pi}\int_0^3 f(r\cos(\vartheta), r\sin(\vartheta))rdrd\vartheta$$

Example 1.3. Evaluate $\int\int_R (x+y)dA$ where R is the second region illustrated in the last example.

We get

$$\int\int_R f(x,y)dA = \int_0^{\pi}\int_0^3 f(r\cos(\vartheta), r\sin(\vartheta))rdrd\vartheta = \int_0^{\pi}\int_0^3 r^2\cos\vartheta + r^2\sin(\vartheta)drd\vartheta$$

$$= \int_0^{\pi} \frac{r^3}{3}(\cos\vartheta + \sin(\vartheta))\Big|_0^3 d\vartheta = 9\int_0^{\pi} cos\vartheta + \sin(\vartheta)d\vartheta = 9\Big[\sin\vartheta - \cos(\vartheta)\Big]_0^{\pi} = 18$$

2. Polar Integrals over Non-Rectangular Regions

Just as with Cartesian coordinates, we can modify the limits over a polar integral to integrate over a region which is not a polar rectangle. As with the Cartesian case, it is important to observe that the limits in this case will depend upon a certain variable. Usually we only consider r as a function of ϑ in the limits, though in principal we could also consider ϑ as a function of r (though this is usually much more difficult). In general we get the following:

Result 2.1. If f is continuous on a polar region of the form $\alpha \leqslant \vartheta \leqslant \beta$ and $h_1(\vartheta) \leqslant r \leqslant h_2(\vartheta)$, then

$$\int\int_R f(x,y)dA = \int_{\alpha}^{\beta}\int_{h_1(\vartheta)}^{h_2(\vartheta)} f(r\cos(\vartheta), r\sin(\vartheta))rdrd\vartheta$$

We illustrate with a couple of examples.

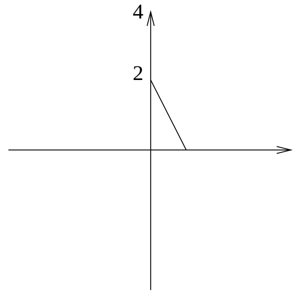
Example 2.2. Find the area of the region enclosed by one leaf of the rose $r = \cos(\vartheta)$.

Observe that one leaf occurs with $-\pi/6 \leqslant \vartheta \leqslant \pi/6$. We already know r is bounded below by 0 and above by $\cos 3\vartheta$, so to find the area we integrate the constant function 1 over this region. We get:

$$\int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\vartheta} 1rdrd\vartheta = \int_{-\pi/6}^{\pi/6}\cos^2(3\vartheta)d\vartheta = \left[\frac{\vartheta}{4} + \frac{1}{12}\sin(6\vartheta)\right]_{-\pi/6}^{\pi/6} = \frac{\pi}{12}$$

Example 2.3. Set up integrals for $f(x, y)$ in polar coordinates over the following regions:

(i) The region bounded by the both axes and the line $f(x) = -2x + 2$ illustrated below:



We have $0 \leqslant \vartheta \leqslant \pi/2$, so we just need to determine how r depends upon ϑ. Clearly we have $0 \leqslant r$, and the largest value of r depends upon ϑ. Since $x = r\cos(\vartheta)$ and $y = r\sin\vartheta$, putting them into the equation for the line, we have $r\sin\vartheta = -2r\cos(\vartheta) + 2$. Solving for r, we get

$$r = \frac{2}{\sin\vartheta + 2\cos\vartheta}.$$

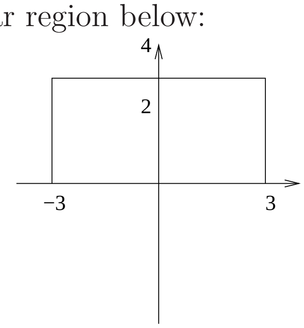
Thus we have

$$0 \leqslant r \leqslant \frac{2}{\sin\vartheta + 2\cos\vartheta}.$$

Hence

$$\int\int_R f(x,y)dA = \int_0^{\pi/2}\int_0^{\frac{2}{\sin\vartheta+2\cos\vartheta}} f(r\cos(\vartheta), r\sin(\vartheta))rdrd\vartheta.$$

(ii) The rectangular region below:



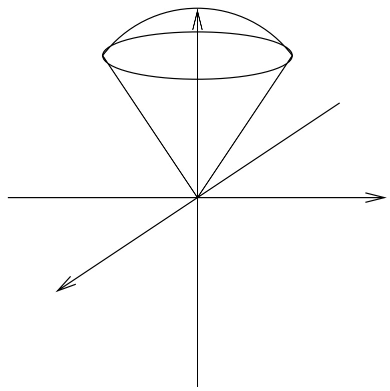
We need to break this integral up into three different pieces. For $0 \leqslant \vartheta \leqslant \pi/4$, we have x bounded between 0 and 2 meaning $0 \leqslant r \leqslant 3/\cos(\vartheta)$ For $\pi/2 \leqslant \vartheta \leqslant 3\pi/2$, we have $0 \leqslant r \leqslant 2/\sin\vartheta$ since the region is bounded by $y = 0$ and $y = 2$, and for $3\pi/2 \leqslant \vartheta \leqslant \pi$, we have $0 \leqslant r \leqslant -3/\cos\vartheta$ since the region is bounded by $x = 0$ and $x = -3$. Therefore, we get

$$\int\int_R f(x,y)dA = \int_0^{\pi/2}\int_0^{2/\cos\vartheta} f(r\cos(\vartheta), r\sin(\vartheta))rdrd\vartheta +$$

$$\int_{\pi/2}^{3\pi/2}\int_0^{2/\sin\vartheta} f(r\cos(\vartheta), r\sin(\vartheta))r\,dr\,d\vartheta + \int_{3\pi/2}^{\pi}\int_0^{-2/\cos\vartheta} f(r\cos(\vartheta), r\sin(\vartheta))r\,dr\,d\vartheta$$

Example 2.4. Find the volume of the region bounded below $z = \sqrt{x^2+y^2}$ and above by $x^2+y^2+z^2=1$.

First observe that the region looks like the following:



To find the volume under a function, we need to evaluate the integral over the corresponding region. Therefore, the first thing we need to do is evaluate what 2d-region this volume is above. Since $x^2+y^2+z^2=1$ and $z=\sqrt{x^2+y^2}$, it follows that $x^2+y^2+x^2+y^2=1$, so the region in the xy-plane below this volume is the circle $x^2+y^2=1/2$. Therefore to find the volume, we need to integrate

$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2})r\,dr\,d\vartheta = \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}(\sqrt{1-r^2}-\sqrt{r^2})r\,dr\,d\vartheta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}-r^2\,dr\,d\vartheta = 2\pi\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}-r^2\,dr$$

$$= 2\pi\left[-\frac{(1-r^2)^{3/2}}{3}-\frac{r^3}{3}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{2\pi}{3}\left[1-\frac{1}{\sqrt{2}}\right]$$