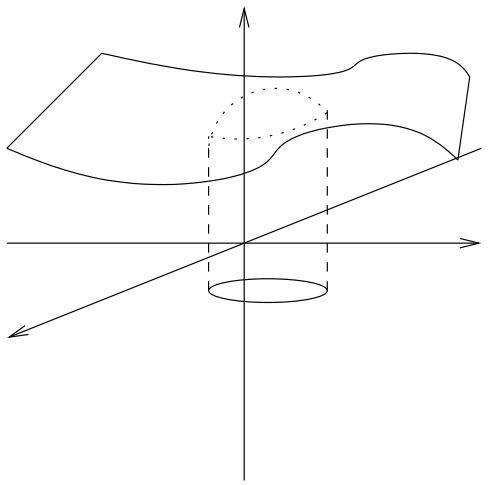
Section 16.5
Applications of Integration

"Using Double Integrals to find Surface Area and other Applications"

In Calculus 2, we used integrals to find the surface area of a surface of revolution. Though this method could be used to find the surface area of the sphere, a paraboloid and some other surfaces, the fact that it only works for surfaces which are revolutions about some axis means that it cannot be used to find the areas of most surfaces. In this section, we use double integrals to determine a formula for surface area of any surface (provided the defining equations are nice enough).

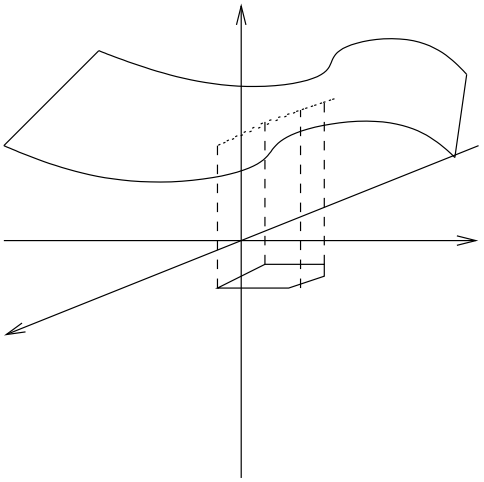
1. Surface Area

Suppose that S is a surface which is the graph of some function $z = f(x, y)$ which has continuous partial derivatives and suppose we want to find the surface area of $f(x, y)$ over some region R in the plane (see illustration).



We proceed as follows.

- (i) We break up the region R into small rectangles with side lengths of Δx and Δy. We ignore any rectangles which are not fully contained in R (we will be taking a sum as Δx and Δy go to zero).
- (ii) Let the point $(\bar{x}_i, \bar{y}_j)$ denote the point in he bottom left hand corner of each rectangle (we could choose any point, but we choose this one for convenience).
- (iii) Observe that the surface area of the region of the surface above the ijth rectangle can be approximated by the area on the tangent plane above the rectangle to the surface at the point $\bar{x}_i, \bar{y}_j)$ (see illustration). Therefore we need to find the area of this parallelogram.



(*iv*) Let $\vec{u}$ and $\vec{v}$ denote the vectors along the edges of the parallelogram where $\vec{u}$ is in the x-direction and $\vec{v}$ is in the y direction. We observe that the area of the parallelogram will be the magnitude of the cross-product $||\vec{u} \times \vec{v}||$ (WHY?). Therefore, we just need to find $\vec{u}$ and $\vec{v}$.

(*v*) Observe that $\vec{u}$ moves a distance Δx in the x-direction, and a distance $f_x(\bar{x}_i, \bar{y}_j)\Delta x$ in the z-direction and similar observations hold for y. In particular, we have $\vec{u} = \Delta x\vec{i} + (f_x(\bar{x}_i, \bar{y}_j)\Delta x)\vec{k}$ and $\vec{v} = \Delta x\vec{j} + (f_y(\bar{x}_i, \bar{y}_j)\Delta y)\vec{k}$.

(*vi*) This means the area of the parallelogram will be

$$||\vec{u} \times \vec{v}|| = ||(\Delta x\vec{i} + (f_x(\bar{x}_i, \bar{y}_j)\Delta x)\vec{k}) \times (\Delta y\vec{j} + (f_y(\bar{x}_i, \bar{y}_j)\Delta y)\vec{k})||$$

$$= \sqrt{(\Delta x\Delta y)^2 + (f_y\Delta y\Delta x)^2 + (f_x\Delta x\Delta y)^2} = \sqrt{1 + (f_y)^2 + (f_x)^2}\Delta x\Delta y$$

(*vii*) Summing over all the rectangles, we can approximate the surface area by

$$S \cong \sum_{j=1}^{m}\sum_{i=1}^{n} \sqrt{1 + (f_y(\bar{x}_i, \bar{y}_j))^2 + (f_x(\bar{x}_i, \bar{y}_j))^2}\Delta x\Delta y.$$

Letting $m, n \to \infty$, this approximation gets closer and closer to the actual value.

Putting these facts together, we get the following general formula for the surface area for the graph of a function of two variables.

Result 1.1. If $f(x, y)$ is a continuous function and R is some region in the plane, then the surface area of $f(x, y)$ over the region R is given by the formula

$$A(S) = \int\int_R \sqrt{1 + f_x^2 + f_y^2}dA$$

Calculations are straight forward.

Example 1.2. (i) Determine the surface area for a sphere of radius 1.

We can determine the surface area for half a sphere and then multiply by 2. A defining function for the sphere is $f(x,y) = \sqrt{1-x^2-y^2}$, so we want to integrate the function $\sqrt{1+f_x^2+f_y^2}$ over the unit disc $x^2+y^2 \leqslant 1$. Observe that

$$f_x = -\frac{2x}{\sqrt{1-x^2-y^2}}$$

and similarly for y, so we want to integrate the function

$$\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}.$$

Since this is a circular region, we shall convert to polars, so we have:

$$\int\int_R \frac{1}{\sqrt{1-x^2-y^2}}dA = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}rdrd\vartheta = \int_0^{2\pi} -\sqrt{1-r^2}\Big|_0^1 d\vartheta$$
$$= \int_0^1 1d\vartheta = 2\pi$$

Hence the surface area of the unit sphere is 4π which agrees with the regular formula $S = 4\pi r^2$.

(ii) Find the surface area of the part of the surface $z = xy$ which lies within the cylinder $x^2+y^2 = 1$.

We have $f_x = y$ and $f_y = x$ and we are integrating over the circle $x^2+y^2 = 1$. thus we need to calculate

$$\int\int_R \sqrt{1+x^2+y^2}dA$$

Since it is a circular region, we shall use polars; in particular, we have

$$\int\int_R \sqrt{1+x^2+y^2}dA = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}rdrd\vartheta = 2\pi\left[-\frac{(1-r^2)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{2\pi}{3}$$

Other examples:

Example 1.3. Find the surface area of the following:

(i) The part of the plane $3x+2y+z = 6$ that lies in the first octant.

(ii) The part of the cylinder $y^2+z^2 = 1$ that lies above the rectangle with vertices $(0,0)$, $(0,1)$ and $(2,1)$.

(iii) The part of the sphere $x^2+y^2+z^2 = 4z$ which lies inside the paraboliod $z = x^2+y^2$.