Section 16.6
Triple Integrals

"Integrating Functions of three Variables"

In this section we consider integrating functions of three variables over a region in 3-space. We shall be using the ideas developed for a function of two variables, so we shall omit most of the proofs and explanations referring to the function of two variables case. The biggest difference in this case is that there is not really a very viable geometric description of an integral of a function of three variables - it does not represent volume or area, but a 4-dimensional analogue of this (sometimes density or time).

1. Triple Integrals over Boxes

Like with double integrals, we start by looking at integrals over fairly easy regions. Suppose that B is a box in 3-space given by $[a, b] \times [c, d] \times [e, f]$ and $f(x, y, z)$ is continuous on B. Then we define a triple integral as follows:

Definition 1.1.

$$\int\int\int f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_j\bar{z}_k)\Delta x\Delta y\Delta z$$

where $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ are fixed points in the ijk-th subbox after we subdivide B into subboxes of side lengths $\Delta x = (b-a)/n$, $\Delta y = (d-c)/m$ and $(f-e)/l = \Delta z$ provided this limit exists.

Just as with double integrals, when evaluating a triple integral, we can use Fubini's Theorem.

Result 1.2.

$$\int\int\int f(x,y,z)dV = \int_e^f\int_c^d\int_a^b f(x,y,z)dxdydz$$

(or any other order).

We illustrate with an example.

Example 1.3. Evaluate the triple integral

$$\int\int\int_B (xy+z)dV$$

where $B = [0, 1] \times [0, 1] \times [-1, 1]$.
To do this, we use Fubini's Theorem.

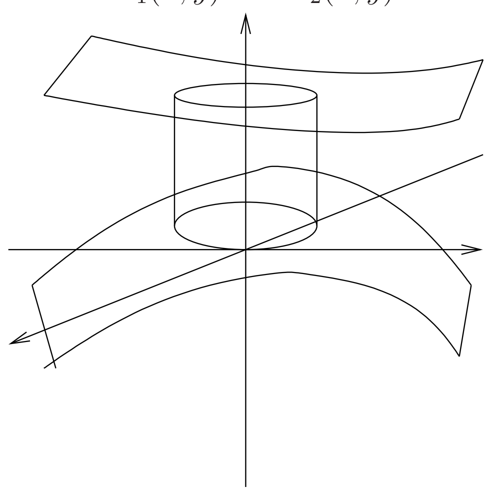
$$\int\int\int_B (xy+z)dV = \int_{-1}^1\int_0^1\int_0^1 (xy+z)dxdydz = \int_{-1}^1\int_0^1 \left[\frac{x^2y}{2}+zx\right]_0^1 dydz$$

$$= \int_{-1}^{1}\int_{0}^{1}(\frac{y}{2}+z)dydz = \int_{-1}^{1}\left[\frac{y^2}{4}+zy\right]_0^1 dz = \int_{-1}^{1}(\frac{1}{4}+z)dz = \left[\frac{z}{4}+\frac{z^2}{2}\right]_{-1}^{1} = \frac{1}{2}$$

2. Triple Integrals over General Regions

The real difficulty of triple integrals arises when we are integrating over regions which are not boxes (as with double integrals). Similar to double integrals, we shall break triple integrals into three different cases. Remember that since we already know how to evaluate a double integral, the only part of the iterated integral we need to learn how to evaluate is the first of the three integrals (because once this is evaluated, we are left with a double integral).

2.1. **Triple Integrals of Type 1.** Suppose that R is some general region in 3-space. We say it is of type 1 if it lies between the graphs of two continuous functions $u_1(x, y)$ and $u_2(x, y)$ as illustrated below.



Such a region is described by

$$\{(x, y, z)|(x, y) \in D, u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of R onto the xy-plane. Therefore, the upper z value in R foir a fixed value of x and y in D is $u_2(x, y)$ and the lower limit is $u_1(x, y)$. This means by a similar argument to the function of two variables case, if R is of type 1, then it can be evaluated as follows:

Result 2.1.

$$\int\int\int_R f(x, y, z)dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right] dA$$

where D is the **projection** of R in the xy-plane.

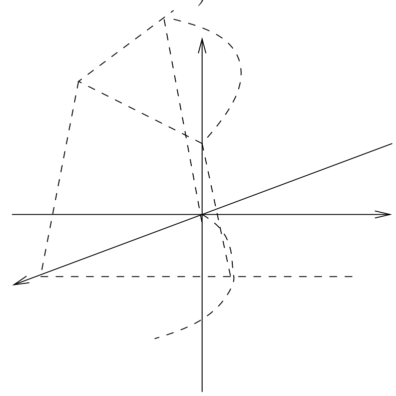
Observe that the remaining integral is a double integral in the plane, so we can use the methods we developed in the previous sections to evaluate this integral as a type 1 or type 2 double integral. We illustrate with some examples.

Example 2.2. (*i*) Evaluate the integral

$$\int\int\int_R 6xydV$$

where R lies under the plane $z = 1+x+y$ and above the region in the xy-plane bounded by the curves $z = \sqrt{x}$ and $x = 1$.

This is a triple integral of type 1 with $0 \leqslant z \leqslant 1 + x + y$. The double integral is of type 1 as well with $0 \leqslant y \leqslant \sqrt{x}$ and $0 \leqslant x \leqslant 1$ (see illustration).



$$\int\int\int_R 6xydV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} (6xy)dzdydx = \int_0^1 \int_0^{\sqrt{x}} (6xy+6x^2y+6xy^2)dydx$$

$$= \int_0^1 \left[3xy^2+3x^2y^2+2xy^3\right]_0^{\sqrt{x}} dx = \int_0^1 (3x^2+3x^3+2x^{\frac{5}{2}})dx = \left[x^3+\frac{3x^4}{4}+\frac{4x^{\frac{7}{2}}}{7}\right]_0^1 = \frac{65}{28}$$

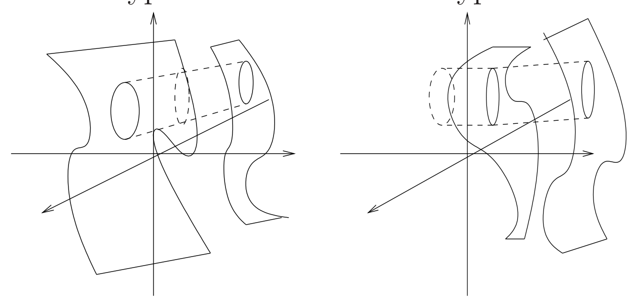
(*ii*) Set up an integral of $f(x, y, z)$ over the ice cream cone bounded between the unit sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

This is a triple integral of type 1. Observe that $\sqrt{x^2 + y^2} \leqslant z \leqslant \sqrt{1 - x^2 - y^2}$. To find the projection of the region in the xy-plane, we observe that it must be bounded by the projection of the intersections of the two curves i.e the points of the sphere where $z = \sqrt{x^2 + y^2}$. Plugging into the equation for the sphere, we have $x^2 + y^2 + x^2 + y^2 = 2$, or $x^2 + y^2 = 1/2$. Thus the projection is a circle of radius $1/\sqrt{2}$. This can be interpreted as an integral of type 2 or 1. We use type 2 giving $-\sqrt{1/2 - y^2} \leqslant x \leqslant \sqrt{1/2 - y^2}$ and $-1/\sqrt{2} \leqslant y \leqslant 1/\sqrt{2}$, so we get

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-y^2}}^{\sqrt{\frac{1}{2}-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z)dzdxdy$$

2.2. **Triple Integrals of Type 2 and 3.** Integrals of type 2 and 3 are similar to type 1, the only difference being that the region is bounded by either functions of y and z (Type 2) or x and z (Type 3), see the illustrations below.

Type 2 Type 3



$h_1(y,z) \leqslant x \leqslant h_2(y,z)$ $g_1(x,z) \leqslant y \leqslant g_2(x,z)$

Because they are all very similar, we illustrate with just a single example of Type 2.

Example 2.3. Evaluate

$$\int\int\int_R x dV$$

where R is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

Here we have $4y^2 + 4z^2 \leqslant x \leqslant 4$, and the projection of the region into the yz-plane is the circle $4y^2 + 4z^2 = 4$ or $y^2 + z^2 = 1$. We can bound this as $-\sqrt{1-y^2} \leqslant z \leqslant \sqrt{1-y^2}$ and $-1 \leqslant y \leqslant 1$. Thus we have

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4y^2+4z^2}^{4} x dx dz dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x^2}{2}\bigg|_{4y^2+4z^2}^{4} dz dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 8 - \frac{(4y^2+4z^2)^2}{2} dz dy$$

Since this is a circular region, we can use a polar integral. Specifically, we use $y = r\cos(\vartheta)$ and $z = r\sin(\vartheta)$, so

$$\int_0^{2\pi}\int_0^1 \left(8 - \frac{(4r^2)^2}{2}\right) r dr d\vartheta dz d = \int_0^{2\pi}\int_0^1 (8r - 8r^5) dr d\vartheta$$

$$= 2\pi\left[4r^2 - \frac{4r^6}{3}\right]_0^1 = 2\pi\frac{8}{3} = \frac{16\pi}{3}$$

3. Applications of Triple Integrals

We finish by considering a couple of applications of triple integrals. First we observe that the integral of the constant function $f(x,y,z) = 1$ will give the volume of the region we are integrating over. We illustrate.

Example 3.1. Find the volume of the solid bounded by the surface $y = x^2$ and the planes $z = 0$, $z = 4$ and $y = 9$.

We need to evaluate

$$\int\int\int_r 1dV$$

where R is the region described. Over this region, we have $-3 \leqslant x \leqslant 3$, $x^2 \leqslant y \leqslant 9$ and $0 \leqslant z \leqslant 4$. Thus we get

$$V = \int_0^4 \int_{-3}^3 \int_{x^2}^9 1dydxdz = \int_0^4 \int_{-3}^3 (9-x^2)dxdz = 4\left[9x - \frac{x^3}{3}\Big|_{-3}^3\right] = 144$$

One practical application of integration of a function of three variables is finding the mass. If the density of a solid S is given by $\varrho(x, y, z)$, then the mass of the solid can be calculated by integrating the density function over the solid. We illustrate.

Example 3.2. Suppose S is the cube with side lengths 2 centered at the origin. If the density of the cube is $\varrho(x, y, z) = x^2+y^2+z^2$, calculate the mass of the cube.

Observe that this function is symmetric across all eight octants, so we just need to find

$$8\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)dxdydz = 8\int_0^1 \int_0^1 (\frac{x^3}{3} + y^2x + z^2x\Big|_0^1)dydz$$
$$= 8\int_0^1 \int_0^1 (\frac{1}{3} + y^2 + z^2)dydz = 8\int_0^1 \int_0^1 (\frac{y}{3} + \frac{y^3}{3} + z^2y\Big|_0^1)dz$$
$$= 8\int_0^1 (\frac{2}{3} + z^2)dz = 8(\frac{2}{3} + \frac{z^3}{3}\Big|_0^1) = \frac{16}{3}$$

Other Examples:

(i) Set up a triple integral for a function $f(x, y, z)$ over the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

(ii) Set up two triple integrals of $f(x, y, z)$ over the cylinder $x^2 + y^2 \leqslant 1$ using Cartesian coordinates for the first and then using polar coordinates for the second.