

Section 16.7 Triple Integrals in Cylindrical Coordinates

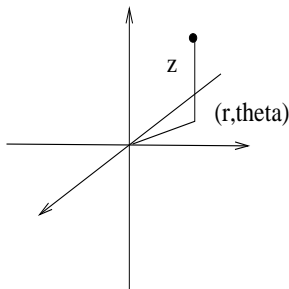
“Integrating Functions in Different Coordinate Systems”

In Section 16.4, we used the polar coordinate system to help integrate functions over circular regions. In the next two sections, we consider two new coordinate systems to help integrate functions over particular types of region in 3-space. We shall first consider a generalization to polar coordinates.

1. CYLINDRICAL COORDINATES

The first coordinate system we consider is a generalization of polar coordinates - the basic idea is to take the polar coordinates in the xy -plane and then simply add the z -coordinate to determine the height of a point. They are particularly useful when describing cylinders. Formally, we define the cylindrical coordinate system as follows.

Definition 1.1. The cylindrical coordinates of a point P in 3-space is defined to be (r, ϑ, z) where (r, ϑ) are the polar coordinates of the projection of P in the xy -plane and z is the z -coordinate of the plane where $r \geq 0$ and $0 \leq \vartheta < 2\pi$.

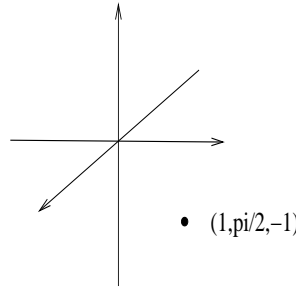


Since cylindrical coordinates are so closely related to polar coordinates, it is easy to convert from rectangular coordinates in 3-space into cylindrical and vice versa.

- Result 1.2.**
- (i) The rectangular coordinates of the point (r, ϑ, z) in 3 space are $x = r \cos(\vartheta)$, $y = r \sin(\vartheta)$ and $z = z$.
 - (ii) The cylindrical coordinates of the point (x, y, z) can be found by solving the equations $x^2 + y^2 = r^2$, $\tan(\vartheta) = y/x$ and $z = z$.

Remember that when converting from rectangular to polar, you need to be very careful with the angle ϑ because simply applying the inverse of $\tan(x)$ on a calculator will not necessarily give you the correct answer (WHY?). We illustrate with a few examples.

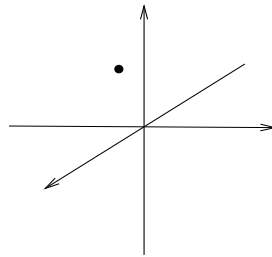
Example 1.3. Plot the point with cylindrical coordinates $(1, \pi/2, -1)$ and convert to rectangular coordinates.



Converting, we have $x = 0$, $y = 1$ and $z = -1$, so $(0, 1, -1)$.

Example 1.4. Plot the point with rectangular coordinates $(-1, -2, 4)$ and convert to cylindrical coordinates.

$(-1, -1, 4)$



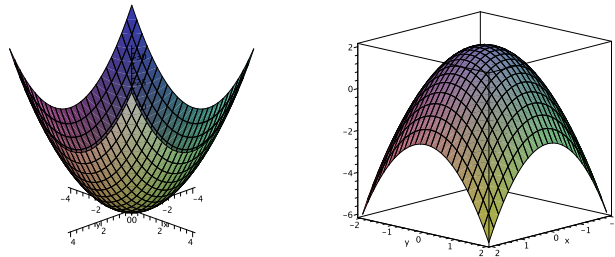
Converting, we have $r^2 = 1 + 1 = 2$, so $r = \sqrt{2}$, $\tan(\vartheta) = 1$, so $\vartheta = \pi/4$ or $3\pi/4$ and $z = 4$. To determine ϑ , we just observe in the diagram it cannot be $\pi/4$, so the coordinates are $(1, 3\pi/4, 4)$.

Example 1.5. Describe the surface $r^2 + z^2 = 1$.

Substituting $r^2 = x^2 + y^2$, we see that this is simply the equation for a sphere of radius 1 centered at the origin.

Example 1.6. Sketch the region bounded by $r^2 \leq z \leq 2 - r^2$

To do this, we first set the equations equal giving the equations $r^2 = z$ and $z = 2 - r^2$. Both of these are equations for elliptic paraboloids, one with base at the origin centered in the z direction pointing in the negative z -direction, and the other has its base in the z -axis at $z = 2$ and is pointing upward (this can be seen by simply substituting in x and y). The region we will be looking for is the region between these two paraboloids



Example 1.7. Write the equation $y = 5$ in polar coordinates.

We know $y = r \sin(\vartheta)$, so we must have $r \sin(\vartheta) = 5$ as the polar equation for this rectangular function (there are no restrictions on z or x).

2. TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Suppose that R is a region in 3-space of type 1, so it is a region whose z values are bounded by continuous functions of x and y - that is

$$R = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection in the xy -plane. Then from the previous section, we will have

$$\int \int \int_R f(x, y, z) dV = \int \int_D \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dA.$$

If the region D is a polar region, the last two integrals can be evaluated as a polar integral, and in fact since $x = r \cos(\vartheta)$ and $y = r \sin(\vartheta)$, the whole integral can be rewritten as a polar integral by converting to polar coordinates. Specifically, if the region D can be converted to a polar region with $h_1(\vartheta) \leq r \leq h_2(\vartheta)$ and $\vartheta_1 \leq \vartheta \leq \vartheta_2$, then we will have:

$$\begin{aligned} & \int \int \int_R f(x, y, z) dV \\ &= \int_{\vartheta_1}^{\vartheta_2} \int_{h_1(\vartheta)}^{h_2(\vartheta)} \int_{u_1(r \cos(\vartheta), r \sin(\vartheta))}^{u_2(r \cos(\vartheta), r \sin(\vartheta))} f(r \cos(\vartheta), r \sin(\vartheta), z) r dz dr d\vartheta. \end{aligned}$$

Thus to evaluate a triple integral in cylindrical coordinates, we do the following:

- (i) Convert the function $f(x, y, z)$ into a cylindrical function.
- (ii) Convert the projection D into a polar region.
- (iii) Change the limits of the integral and include the “ r ” in the integral.
- (iv) Evaluate.

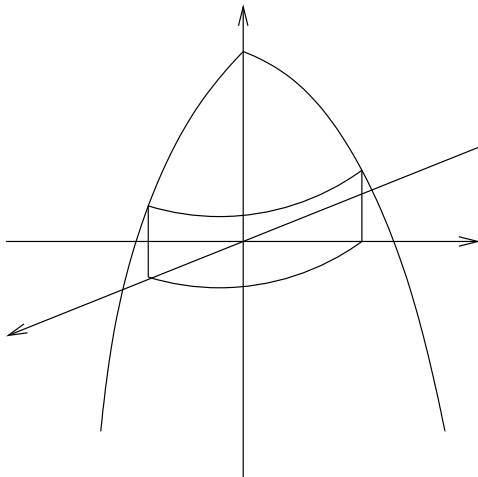
We illustrate with some examples.

Example 2.1. Sketch the cylindrical region over which the following integral is being performed:

$$\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r dz dr d\vartheta$$

Here we have $0 \leq \vartheta \leq \pi/2$, $0 \leq r \leq 2$ and $0 \leq z \leq 9 - r^2 = 9 - x^2 - y^2$. The inequality on ϑ restricts to the first quadrant of the xy -plane and the inequality on r restricts to radial values from the origin between 0 and 2. With the last inequality, we observe that the equation $z =$

$9 - x^2 - y^2$ is a parabolic bowl which has been inverted and moved up a distance 9. This the region of integration is the first quadrant with $0 \leq r \leq 2$ which lies above the xy -plane and below the inverted parabolic bowl $z = 9 - x^2 - y^2$.



Example 2.2. Find the volume of the region which lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

We need to evaluate the integral

$$\int \int \int_R 1 dV$$

where R is the region in question. Converting the to cylindrical coordinates, we have $0 \leq \vartheta \leq 2\pi$, $0 \leq r \leq 1$ and $-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$. By symmetry, we can just integrate over the top part, so we need to find

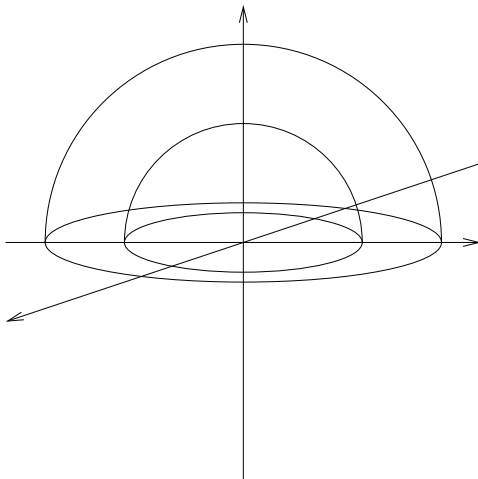
$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r dz dr d\vartheta &= \int_0^{2\pi} \int_0^1 (r\sqrt{4-r^2}) dr d\vartheta \\ &= \int_0^{2\pi} \left(-\frac{(4-r^2)^{\frac{3}{2}}}{3} \right) \Big|_0^1 d\vartheta = 2\pi \left[\left(-\frac{3^{\frac{3}{2}}}{3} \right) - \left(-\frac{4^{\frac{3}{2}}}{3} \right) \right] = 2\pi \left[\frac{8}{3} - \sqrt{3} \right]. \end{aligned}$$

Thus the volume is

$$V = 4\pi \left[\frac{8}{3} - \sqrt{3} \right]$$

Example 2.3. Set up and evaluate a cylindrical integral of $f(x, y, z) = 2z$ over the upper shell (so $z \geq 0$) bounded between the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

The region looks like the following:



Observe that over the circle $x^2 + y^2 = 1$, the values of z are bounded below by the inner sphere and above by the outer sphere, so $\sqrt{1 - r^2} \leq z \leq \sqrt{4 - r^2}$. In the ring between the circle $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, the z values are bounded below by 0 and above by the outer sphere, so $0 \leq z \leq \sqrt{4 - r^2}$. This means to evaluate the integral in cylindrical coordinates, we need to use two integrals depending upon which polar region we are in. Specifically, we have

$$\begin{aligned}
 \int_R 2z dV &= \int_0^{2\pi} \int_0^1 \int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} (2z) r dz dr d\vartheta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{4-r^2}} (2z) r dz dr d\vartheta \\
 &= \int_0^{2\pi} \int_0^1 (z^2 r) \Big|_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} dr d\vartheta + \int_0^{2\pi} \int_1^2 (z^2 r) \Big|_0^{\sqrt{4-r^2}} dr d\vartheta \\
 &= \int_0^{2\pi} \int_0^1 r(4 - r^2) - r(1 - r^2) dr d\vartheta + \int_0^{2\pi} \int_1^2 r(4 - r^2) dr d\vartheta \\
 &= \int_0^{2\pi} \int_0^1 3r dr d\vartheta + \int_0^{2\pi} \int_1^2 4r - r^3 dr d\vartheta \\
 &= \int_0^{2\pi} \frac{3r^2}{2} \Big|_0^1 d\vartheta + \int_0^{2\pi} 2r^2 - \frac{r^4}{4} \Big|_1^2 d\vartheta = \int_0^{2\pi} \frac{15}{4} d\vartheta = \frac{15\pi}{2}
 \end{aligned}$$

Notice that for this last integral, cylindrical coordinates did not really make this much easier. This motivates our next topic of study - evaluating integrals using a system of coordinates called cylindrical coordinates.